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# RESOLUTIONS BY POLYGRAPHS

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ABSTRACT. A notion of resolution for higher-dimensional categories is defined, by using polygraphs, and basic invariance theorems are proved.

## 1. Introduction

Higher-dimensional categories naturally appear in the study of various rewriting systems. A very simple example is the presentation of $\mathbf{Z}/2\mathbf{Z}$ by a generator $a$ and the relation $aa \to 1$. These data build a 2-category $X$ :

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} X_2$$

where $X_0 = \{\bullet\}$ has a unique 0-cell, $X_1 = \{a^n/n \geq 0\}$ and $X_2$ consists of 2-cells $a^n \to a^p$, corresponding to different ways of rewriting $a^n$ to $a^p$ by repetitions of $aa \to 1$, up to suitable identifications. 1-cells compose according to $a^n *_0 a^p = a^{n+p}$, and 2-cells compose vertically, as well as horizontally, as shown on Figure 1, whence the 2-categorical structure on $X$.

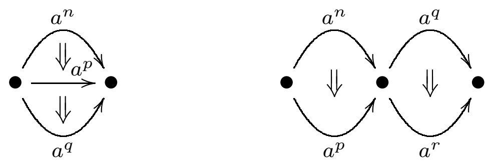


Figure 1: composition of 2-cells

In this setting, we recover the original monoidal structure of $\mathbf{Z}/2\mathbf{Z}$ by collapsing the 2-cells (see [4]) to identities. Likewise, tree-rewriting systems could be expressed in the framework of 3-categories. Thus we start from the fact that many structures of interest are in fact $n$-categories, while computations in these structures take place in $(n+1)$-categories.

On the other hand, if a monoid can be presented by a finite, noetherian and confluent rewriting system, then its homology groups are finitely generated, as proved by Squier

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and others (see [15, 1, 11, 13]). Precisely, it is shown how to build a free resolution of $\mathbf{Z}$ by $\mathbf{Z}M$-modules by using a complete (i.e. finite, noetherian and confluent) presentation $(\Sigma, R)$ of the monoid $M$. Subsequently, the notion of *derivation type* was introduced: roughly speaking, one asks if the equivalence between rewriting paths with the same source and target can be generated by a finite number of basic deformations. It turns out that this property is independent of the (finite) presentation chosen, and that it implies homological finiteness in dimension 3 (see [16, 12, 8]).

As regards $\infty$-categories, homology can be defined through simplicial nerves (see [17, 6]). However, this does not lead to easy computations. What we would like to do is to extract structural invariants from particular presentations. The present work is a first step in this vast program: by using Burroni's idea of polygraph, we propose a definition of *resolution* for $\infty$-categories, which can be seen as a non-commutative analogue of a free resolution in homological algebra. These new resolutions generalize in arbitrary dimensions the constructions which already appear in the study of derivation types.

The main result (Theorem 5.1) establishes a strong equivalence between any two resolutions of the same category, and yields homological invariance as a consequence (Theorem 6.1).

All the material we present here was elaborated in collaboration with Albert Burroni, whose companion paper [5] will be released very soon.

## 2. Graphs and Categories

2.1. Basic definitions. Polygraphs have been defined by Burroni in [4]. Let us briefly recall the main steps of his construction: a *graph* $X$ is given by a diagram

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1$$

$X_0$ is the set of vertices–or 0-cells–$X_1$ the set of oriented edges–or 1-cells–and the applications $s_0$, $t_0$ are respectively the source and target. We usually denote $x^1 : x^0 \to y^0$ in case the 1-cell $x^1$ has source $x^0$ and target $y^0$. More generally, an *$n$-graph*, also known as *globular set* will be defined by a diagram

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} \quad \cdots \quad \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} X_n \tag{1}$$

with relations

$$s_i s_{i+1} = s_i t_{i+1} \qquad t_i s_{i+1} = t_i t_{i+1}$$

Figure 2 conveys the geometric meaning of these equations.

If diagram (1) is not bounded on the right, we get an $\infty$-graph. Elements of $X_i$ are called $i$-cells and will be denoted by $x^i$. By

$$x^{i+1} : x^i_0 \to x^i_1$$

$$
\begin{tikzcd}
ssx \arrow[r, bend left, "sx"] \arrow[r, bend right, "tx"'] \arrow[r, phantom, "\Downarrow x"] & ttx
\end{tikzcd}
$$

Figure 2: cell

we mean that $x^{i+1}$ has $i$-dimensional source $x_0^i$ and target $x_1^i$.

For $i \geq 1$, two $i$-cells $x^i$ and $y^i$ are called *parallel* if they have the same source and the same target. This is denoted by $x^i \parallel y^i$. Also any two 0-cells are considered to be parallel.

A morphism $\phi : X \to Y$ between $n$-graphs $X$ and $Y$ is a family of maps $\phi_i : X_i \to Y_i$ commuting with source and target:

$$
\begin{tikzcd}
X_i \arrow[d, "\phi_i"'] & X_{i+1} \arrow[l, shift right, "s_i"'] \arrow[l, shift left, "t_i"] \arrow[d, "\phi_{i+1}"] \\
Y_i & Y_{i+1} \arrow[l, shift right, "s_i"'] \arrow[l, shift left, "t_i"]
\end{tikzcd}
$$

> *Warning*: for simplicity, in all diagrams, double arrows will stand for source and target maps, except otherwise mentioned. Throughout this paper, the "commutativity" of a diagram containing such arrows means in fact the separate commutativity of two diagrams, obtained by keeping either all source maps or all target maps.

The category of $n$-graphs ($\infty$-graphs) will be denoted by $\mathbf{Grph}_n$ ($\mathbf{Grph}_\infty$). Let $X$ be an $\infty$-graph, and $0 \leq i < j$, put

$$
\begin{aligned}
s_{ij} &= s_i s_{i+1} \dots s_{j-1} \\
t_{ij} &= t_i t_{i+1} \dots t_{j-1}
\end{aligned}
$$

thus defining a new graph $X_{ij}$ for each pair $i < j$:

$$
\begin{tikzcd}
X_i & X_j \arrow[l, shift right, "s_{ij}"'] \arrow[l, shift left, "t_{ij}"]
\end{tikzcd}
$$

The following data determine a structure of $\infty$-category on $X$:

- For all $x^j$, $y^j$ in $X_j$ such that $s_{ij}y^j = t_{ij}x^j$ a composition $x^j *_i y^j \in X_j$:

$$
x^j *_i y^j : \qquad x^i \xrightarrow{x^j} y^i \xrightarrow{y^j} z^i
$$

  We write composition in the order of the arrows, for no reason but personal convenience. For simplicity, the index $j$ does not appear on the symbol $*_i$.

$$\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=3ex, "\Downarrow x^k"] \arrow[r, phantom, shift right=3ex, "\Downarrow y^k"] & \bullet \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=3ex, "\Downarrow u^k"] \arrow[r, phantom, shift right=3ex, "\Downarrow v^k"] & \bullet
\end{tikzcd}$$

Figure 3: exchange

- For each $x^i \in X_i$, a cell $\mathrm{id}^j(x^i) \in X_j$, the $j$-dimensional identity on $x^i$.
- The compositions and identities just defined make each $X_{ij}$ a category.
- The $X_{ij}$'s are compatible, precisely, for each $0 < i < j < k$, $x^k$, $y^k$, $u^k$ and $v^k$

$$(x^k *_j y^k) *_i (u^k *_j v^k) = (x^k *_i u^k) *_j (y^k *_i v^k) \tag{2}$$

provided the first member exists: this is the *exchange rule* (see Figure 3).

Also for all $x^i$,

$$\mathrm{id}^k(x^i) = \mathrm{id}^k(\mathrm{id}^j(x^i)) \tag{3}$$

Finally, for all $i < j < k$ and $i$ composable $j$-cells $x^j$, $y^j$,

$$\mathrm{id}^k(x^j *_i y^j) = \mathrm{id}^k(x^j) *_i \mathrm{id}^k(y^j) \tag{4}$$

These properties allow the following notations:

- For each $j$-cell $x^j$, and $i < j$, $s_{ij}x^j = x_0^i$ and $t_{ij}x^j = x_1^i$.
- In expressions like $x^i *_l y^j$ $(i < j)$, $x^i$ means in fact $\mathrm{id}^j(x^i)$.

Let $i < j$ and $y^j = \mathrm{id}^j(x^i)$, and $e \in \{0, 1\}$. With the previous notations:

$$\begin{array}{rcl}
y_e^k & = & \mathrm{id}^k(x^i) \qquad (i < k < j) \\
 & = & x^i \qquad (k = i) \\
 & = & x_e^k \qquad (0 \leq k < i)
\end{array}$$

The *morphisms* between $\infty$-categories are the morphisms of underlying $\infty$-graphs which preserve the additional structure. Now $\infty$-categories and morphisms build a category $\mathbf{Cat}_\infty$. By restricting the construction to dimensions $\leq n$ we obtain $n$-categories, and the corresponding $\mathbf{Cat}_n$. For each $\infty$-category $X$, we denote by $X^n$ the $n$-category obtained by restriction to cells of dimension $\leq n$.

2.2. Alternative notations. Explicit labels for dimensions make our formulas difficult to read. Hence the need for an alternative notation: let $X$ be an $\infty$-category, the source and target maps from $X_{i+1}$ to $X_i$ will be denoted in each dimension by $\mathrm{d}_-$ and $\mathrm{d}_+$ respectively. If $j = i + k$, $s_{ij}$ and $t_{ij}$ from $X_j$ to $X_i$ will be denoted by $\mathrm{d}_-^k$ and $\mathrm{d}_+^k$, the exponent means here iteration, not dimension.

Likewise, the identity $\mathrm{id}^{i+1}$ from $X_{i+1}$ to $X_i$ will be denoted by I in each dimension, and if $j = i + k$, $\mathrm{id}^j : X_i \to X_j$ becomes $\mathrm{I}^k$. Here again the exponent denotes iteration. Both families of notations will be used freely throughout the paper, sometimes together.

2.3. Example. An important example is the $\infty$-category $BA$ associated to a complex $A$ of abelian groups (see [3], [6]).

$$A_0 \xleftarrow{\partial_0} A_1 \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} A_n \xleftarrow{\partial_n} \cdots$$

The set of $k$-cells is

$$(BA)_k = \prod_{i=0}^{i=k} A_i$$

The source and target maps from $(BA)_{k+1}$ to $(BA)_k$ are given by

$$\begin{aligned} s(a_0, \ldots, a_{k+1}) &= (a_0, \ldots, a_k) \\ t(a_0, \ldots, a_{k+1}) &= (a_0, \ldots, a_k + \partial_k a_{k+1}) \end{aligned}$$

which easily defines an $\infty$-graph. To make it an $\infty$-category, we define the identities by:

$$\mathrm{id}(a_0, \ldots, a_k) = (a_0, \ldots, a_k, 0)$$

and, given $0 \leq j < k$, $\overline{a} = (a_0, \ldots, a_k) \in (BA)_k$ and $\overline{b} = (b_0, \ldots, b_k) \in (BA)_k$ such that $t_{jk}\overline{a} = s_{jk}\overline{b}$, we define their composition along dimension $j$ by:

$$\overline{a} *_j \overline{b} = (a_0, \ldots, a_j, a_{j+1} + b_{j+1}, \ldots, a_k + b_k)$$

We leave the verification of the axioms of $\infty$-categories as an exercise. In fact the category of chain complexes is equivalent to the category of abelian group objects in $\mathbf{Cat}_\infty$.

## 3. Polygraphs

3.1. Formal definition. It is now possible to define *polygraphs*, following [4]. For all $n \geq 0$, there is a category $\mathbf{Cat}_n^+$ given by the pullback:

$$\begin{tikzcd} \mathbf{Cat}_n^+ \arrow[r] \arrow[d] & \mathbf{Grph}_{n+1} \arrow[d, "V_n"] \\ \mathbf{Cat}_n \arrow[r, "U_n"] & \mathbf{Grph}_n \end{tikzcd}$$

$$\begin{tikzcd}
S_0 \arrow[d] & S_1 \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & S_2 \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & S_3 \arrow[dl, shift left] \arrow[dl, shift right] \\
S_0^* & S_1^* \arrow[l, shift left] \arrow[l, shift right] & S_2^* \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

Figure 4: polygraph

where $U$ and $V$ are obvious forgetful functors. An object of $\mathbf{Cat}_n^+$ amounts to an $n$-category with extra $(n+1)$-cells making it an $(n+1)$-graph. We get a forgetful functor $W_{n+1}$ from $\mathbf{Cat}_{n+1}$ to $\mathbf{Cat}_n^+$ factorizing the arrows from $\mathbf{Cat}_{n+1}$ to $\mathbf{Cat}_n$ and $\mathbf{Grph}_{n+1}$:

$$\begin{tikzcd}
\mathbf{Cat}_{n+1} \arrow[dr, "W_{n+1}"] \arrow[drr] \arrow[ddr] & & \\
& \mathbf{Cat}_n^+ \arrow[r] \arrow[d] & \mathbf{Grph}_{n+1} \arrow[d, "V_n"] \\
& \mathbf{Cat}_n \arrow[r, "U_n"] & \mathbf{Grph}_n
\end{tikzcd}$$

It can be shown that $W_{n+1}$ has a left-adjoint $L_{n+1}$. Thus $L_{n+1}(X)$ is the free $(n+1)$-category generated by the $n$-category $X$ with additional $(n+1)$-cells.

An *$n$-polygraph* $S$ is a sequence $S^{(i)}$, where $S^{(0)}$ is a set and for each $1 \leq i \leq n$, $S^{(i)}$ belongs to $\mathbf{Cat}_{i-1}^+$, and built by the following induction process. First chose any set $S_0$ and define $S^{(0)} = S_0$. Suppose $i \geq 1$ and $S^{(i)}$ already defined as an object in $\mathbf{Cat}_{i-1}^+$, or a set if $i = 0$. We get an $i$-category $L_i(S^{(i)})$, and denote by $S_i^*$ the set of its $i$-cells. Then $S^{(i+1)}$ is determined by choosing a set $S_{i+1}$ of $(i+1)$-cells with source and target maps:

$$\begin{tikzcd} S_i^* & S_{i+1} \arrow[l, shift right, "s_i"'] \arrow[l, shift left, "t_i"] \end{tikzcd}$$

defining an $(i+1)$-graph in $L_i(S^{(i)})$. We have just defined an object of $\mathbf{Cat}_i^+$, which completes the induction.

Thus an $n$-polygraph is entirely determined by successive choices of sets $S_i$ together with corresponding source and target maps (Figure 4).

The bottom row in Figure 4 determines the $\infty$-category $S^* = QS$ *generated* by $S$.

Here should be emphasized that the $i$-category $L_i(S^{(i)})$ has the same $k$-cells as $S^{(i)}$ itself for all $k < i$. Moreover let

$$\eta_{S^{(i)}} : S^{(i)} \to W_i L_i S^{(i)}$$

be the unit of the adjunction on the object $S^{(i)}$. It is a morphism in $\mathbf{Cat}_{i-1}^+$ whose components in dimensions $k < i$ are just identities. As for the $i$-component, it is the inclusion of generators:

$$j_i : S_i \to S_i^*$$

These are of course, in each dimension, the vertical arrows on Figure 4.

As a consequence of the above remark, the universal property of polygraphs will be used in the sequel as follows : given a polygraph $S$, an $(n+1)$-category $X$, a family of maps $\phi_i : S_i^* \to X_i$ for $i \in \{0, \ldots, n\}$ defining a morphism in $\mathbf{Cat}_n$, and a map $f_{n+1} : S_{n+1} \to X_{n+1}$ defining together with the $\phi_i$'s a morphism in $\mathbf{Cat}_n^+$, there is a unique $\phi_{n+1} : S_{n+1}^* \to X_{n+1}$ making the family $(\phi_i)_{0 \le i \le n+1}$ a morphism in $\mathbf{Cat}_{n+1}$ and such that the following triangle commutes:

$$\begin{tikzcd} S_{n+1} \arrow[r, "j_{n+1}"] \arrow[dr, "f_{n+1}"'] & S_{n+1}^* \arrow[d, "\phi_{n+1}"] \\ & X_{n+1} \end{tikzcd}$$

A morphism $f$ between polygraphs $S$ and $T$ amounts to a sequence of maps $f_i : S_i \to T_i$ making everything commute. $f$ induces a morphism $Qf$ in $\mathbf{Cat}_\infty$. We get a category **Pol** of polygraphs and morphisms, as well as a functor $Q$:

$$\mathbf{Pol} \xrightarrow{Q} \mathbf{Cat}_\infty$$

3.2. A SMALL EXAMPLE. The 2-category we associated in the introduction to the usual presentation of $\mathbf{Z}/2\mathbf{Z}$ is in fact generated by the following polygraph:

$$\begin{tikzcd} S_0 \arrow[d] & S_1 \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & S_2 \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] \\ S_0^* & S_1^* \arrow[l, shift left] \arrow[l, shift right] & S_2^* \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

with $S_0 = \{\bullet\}$, $S_1 = \{a\}$ where $a$ is a loop:


and $S_2 = \{aa \to 1\}$ where $aa \to 1$ is a 2-cell:

$$\begin{tikzcd} & \bullet \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur, "a"] \arrow[rr, "\mathrm{id}^1(\bullet)"'] & {} & \bullet \end{tikzcd}$$

3.3. LINEARIZATION. Now with each polygraph $S$ we may associate an abelian complex $(ZS, \partial)$: $(ZS)_i$ will be simply the free abelian group $\mathbf{Z}S_i$ on the generators $S_i$. $\partial$ will be defined together with a linearization map $\lambda_i : (QS)_i \to (ZS)_i$ in each dimension such that the following diagram commutes (the unlabeled arrow is of course the inclusion of generators):

$$\begin{tikzcd} S_i \arrow[r, "j_i"] \arrow[dr] & (QS)_i \arrow[d, "\lambda_i"] \\ & (ZS)_i \end{tikzcd}$$

and, for each $x^i \in (QS)_i$:

$$\partial_{i-1}\lambda_i x^i \quad = \quad \lambda_{i-1}t_{i-1}x^i - \lambda_{i-1}s_{i-1}x^i \tag{5}$$

In fact, when representing $x^i$ by an expression built from cells of $S_0, \ldots, S_i$ with compositions and identities, $\lambda_i x^i$ will be simply the linear combination of the non-degenerate $i$-cells occurring in this expression, that is those in $S_i$. In particular, all identities are send to zero.

The idea is to build simultaneously the desired complex $ZS$ and the $\infty$-category $BZS$ (see example 2.3) together with a morphism $QS \to BZS$, by induction on the dimension.

The precise induction hypothesis on $n$ is as follows: for each $i \leq n$ there is a unique map $\lambda_i : (QS)_i \to (ZS)_i$ such that (i) the above triangle commutes (ii) $\partial_i$ defined by (5) makes $(ZS, \partial)$ an $n$-complex, and (iii) the family $(\phi_i)_{0 \leq i \leq n}$ given by

$$\phi_i = \lambda_0 s_{0i} \times \ldots \times \lambda_{i-1}s_{(i-1)i} \times \lambda_i$$

determines a morphism of $n$-categories:

$$\begin{tikzcd} (QS)_0 & (QS)_1 \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right] & (QS)_n \arrow[l, shift left] \arrow[l, shift right] \\ (BZS)_0 \arrow[from=u, "\phi_0"'] & (BZS)_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[from=u, "\phi_1"'] & \cdots \arrow[l, shift left] \arrow[l, shift right] & (BZS)_n \arrow[l, shift left] \arrow[l, shift right] \arrow[from=u, "\phi_n"'] \end{tikzcd}$$

Now the right hand side of (5) is still defined for $i = n+1$ and $x^{n+1} \in S_{n+1}$. We may extend it by linearity to $\mathbf{Z}S_{n+1}$, thus extending $ZS$ to an $(n+1)$-complex. Whence a map $f_{n+1} : S_{n+1} \to (BZS)_{n+1}$ making the following square commutative:

$$\begin{tikzcd} (QS)_n \arrow[d, "\phi_n"'] & S_{n+1} \arrow[l, shift right, "s_n"'] \arrow[l, shift left, "t_n"] \arrow[d, "f_{n+1}"] \\ (BZS)_n & (BZS)_{n+1} \arrow[l, shift right, "s_n"'] \arrow[l, shift left, "t_n"] \end{tikzcd}$$

The universal property of polygraphs then gives a unique $\phi_{n+1} : (QS)_{n+1} \to (BZS)_{n+1}$, which extends the previous $\phi_i's$ to a morphism of $(n+1)$-categories. Let $\pi$ be the projection $(BZS)_{n+1} \to \mathbf{Z}S_{n+1}$, and define $\lambda_{n+1} = \pi\phi_{n+1}$. $\lambda_{n+1}$ satisfies properties (i), (ii) and (iii) in dimension $n+1$, and is the unique such map. Hence the result.

Finally, if $S$ and $T$ are polygraphs, an $u$ is a morphism of $\infty$-categories, $u : QS \to QT$, it has a linearization (Figure 5) from $ZS$ to $ZT$. Precisely, there is a unique $\mathbf{Z}$-linear map $\tilde{u} : ZS \to ZT$, such that for each $x^i \in (QS)_i$,

$$\tilde{u}_i \lambda_i x^i = \lambda_i u_i x^i \tag{6}$$

$$\begin{tikzcd}
S_i \arrow[d] \arrow[dr] & \\
(QS)_i \arrow[r, "\lambda_i"] \arrow[d, "u_i"'] & ZS_i \arrow[d, "\tilde{u}_i"] \\
(QT)_i \arrow[r, "\lambda_i"] & ZT_i \\
T_i \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Figure 5: linearization

In fact, the right member of (6) defines a map $S_i \to ZT_i$, which uniquely extends by linearity to $ZS_i$. Now (6) follows by induction on the complexity of $x^i$.

## 4. Resolutions

We introduce here the central notion of *resolution* for $\infty$-categories. As we shall see, invariants of an $\infty$-category can be computed from a particular resolution, which compares to the role of resolutions in homological algebra.

4.1. Definition. *let $X$ be an $\infty$-category. A* resolution *of $X$ is a pair $(S, \phi)$ where $S$ is a polygraph and $\phi$ is a morphism $QS \to X$ in $\mathbf{Cat}_\infty$ such that*

1. *For all $i \geq 0$, $\phi_i : S_i^* \to X_i$ is surjective.*
2. *For all $i \geq 0$ and all $x^i, y^i \in S_i^*$, if $x^i \parallel y^i$ and $\phi_i x^i = \phi_i y^i$ then there exists $z^{i+1} \in S_{i+1}^*$ such that $z^{i+1} : x^i \to y^i$ and $\phi_{i+1} z^{i+1} = \mathrm{id}^{i+1}(\phi_i x^i)$.*

Intuitively, think of the cells in $S_{i+1}$ as generators for $X_{i+1}$ as well as relations for $X_i$. We refer to the second condition as the *stretching property.* A similar notion appears in [14] where it is called *étirement.*

Likewise an $n$-resolution is given by $\phi : QS \to X$ where $S$ is an $n$-polygraph, and in the Definition 4.1, condition (1) holds for $i \leq n$ and (2) for $i \leq n - 1$.

4.2. Standard resolution. We now turn to a standard resolution for an arbitrary $X$ in $\mathbf{Cat}_\infty$. Let us define an $n$-resolution of $X$ by induction on $n$, in such a way that the $(n+1)$-resolution extends the $n$-resolution.

- For $n = 0$, we simply take $S_0 = X_0$ and $\phi_0$ is the identity map $S_0^* \to X_0$. This is clearly a 0-resolution of $X$.

- Suppose $(S, \phi)$ is an $n$-resolution of $X$, and define $S_{n+1}$ as the set of tuples $c^{n+1} = (z^{n+1}, x^n, y^n)$ where $z^{n+1} \in X_{n+1}$, $x^n, y^n \in S_n^*$, $x^n \parallel y^n$ and $z^{n+1} : \phi_n x^n \to \phi_n y^n$. We may define $f_{n+1} : S_{n+1} \to X_{n+1}$ by $\phi_{n+1} c^{n+1} = z^{n+1}$, and source and target maps $S_{n+1} \to S_n^*$ by $s_n c^{n+1} = x^n$ and $t_n c^{n+1} = y^n$. The universal property of polygraphs gives then $S_{n+1}^*$ and $\phi_{n+1} : S_{n+1}^* \to X_{n+1}$ extending the previous data to an $(n+1)$-resolution of $X$ (see Figure 6).

$$
\begin{tikzcd}
S_n \arrow[d] & S_{n+1} \arrow[dl, shift left] \arrow[dl, shift right] \arrow[d] \arrow[dd, bend left, "f_{n+1}"] \\
S_n^* \arrow[d, "\phi_n"'] & S_{n+1}^* \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\phi_{n+1}"'] \\
X_n & X_{n+1} \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}
$$

Figure 6: universal property

Thus

4.3. Proposition. *Each $\infty$-category $X$ has a resolution.*

The polygraph underlying this standard resolution will be denoted by $PX$, and the corresponding morphism by $\epsilon^X : QPX \to X$.

Now let $u$ be a morphism between $\infty$-categories $X$ and $Y$, we may define a morphism $Pu$ between the polygraphs $PX$ and $PY$, by:

$$
\begin{aligned}
(Pu)_0 &= u_0 && (7)\\
(Pu)_{k+1}(z^{k+1}, x^k, y^k) &= (u_{k+1} z^{k+1}, (QPu)_k x^k, (QPu)_k y^k) && (8)
\end{aligned}
$$

where $(z^{k+1}, x^k, y^k)$ denotes a cell in $(PX)_{k+1}$, that is $z^{k+1} \in X_{k+1}$, $x^k, y^k \in (QPX)_k$ and

$$z^{k+1} : \epsilon_k^X x^k \to \epsilon_k^X y^k$$

The soundness of this definition is shown by considering the diagram:

$$
\begin{tikzcd}
(PX)_{n+1} \arrow[rrr, dotted] \arrow[dd] \arrow[ddr, shift left] \arrow[ddr, shift right] & & & (PY)_{n+1} \arrow[dd] \arrow[ddl, shift left] \arrow[ddl, shift right] \\
& (PX)_n \arrow[r, "(Pu)_n"] \arrow[d] & (PY)_n \arrow[d] & \\
(QPX)_{n+1} \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, "\epsilon^X_{n+1}"'] & (QPX)_n \arrow[r, "(QPu)_n"] \arrow[d, "\epsilon^X_n"'] & (QPY)_n \arrow[d, "\epsilon^Y_n"] & (QPY)_n \arrow[l, shift left] \arrow[l, shift right] \arrow[dd, "\epsilon^Y_{n+1}"] \\
& X_n \arrow[r, "u_n"'] & Y_n & \\
X_{n+1} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rrr, "u_{n+1}"'] & & & Y_{n+1} \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}
$$

We make the induction hypothesis that $Pu$ has been defined up to the dimension $n$ in such a way that the inner squares commute. Then there is a unique arrow from $(PX)_{n+1}$ to $(PY)_{n+1}$ (dotted line) making the whole diagram commutative, and it satisfies (8). Whence a map:

$$(QPX)_{n+1} \xrightarrow{(QPu)_{n+1}} (QPY)_{n+1}$$

such that the following diagram also commutes:

$$
\begin{tikzcd}
(PX)_{n+1} \arrow[r, "(Pu)_{n+1}"] \arrow[d] & (PY)_{n+1} \arrow[d] \\
(QPX)_{n+1} \arrow[r, "(QPu)_{n+1}"] \arrow[d, "\epsilon^X_{n+1}"'] & (QPY)_{n+1} \arrow[d, "\epsilon^X_{n+1}"] \\
X_{n+1} \arrow[r, "u_{n+1}"] & Y_{n+1}
\end{tikzcd}
$$

This gives the required property up to the dimension $n+1$. The uniqueness of the solution easily shows that $P$ is in fact a functor:

$$\mathbf{Cat}_\infty \xrightarrow{P} \mathbf{Pol}$$

and the previous diagrams also show that the arrows

$$QPX \xrightarrow{\epsilon^X} X$$

define a natural transformation $\epsilon : QP \to I$. In fact:

4.4. PROPOSITION. *$Q$ is left-adjoint to $P$.*

Proof. We already have a natural transformation $\epsilon : QP \to I$. It remains to check that each $\epsilon^X$ is universal from $Q$ to $X$, in other words, that for each polygraph $S$ and each $g : QS \to X$ there is a unique $f : S \to PX$ such that the triangle

$$\begin{tikzcd}
& QPX \arrow[d, "\epsilon^X"] \\
QS \arrow[ur, "Qf"] \arrow[r, "g"'] & X
\end{tikzcd}$$

commutes.

We prove by induction on the dimension $k$ that there is a unique family $(f_0, \dots, f_k)$, such that the maps $f_i : S_i \to (PX)_i$ define a morphism of $k$-polygraphs and, for all $0 \leq i \leq k$, the triangle

$$\begin{tikzcd}
& QPX_i \arrow[d, "\epsilon_i^X"] \\
QS_i \arrow[ur, "(Qf)_i"] \arrow[r, "g_i"'] & X
\end{tikzcd}$$

commutes.

For $k = 0$, $QS_0 = S_0$, $(QPX)_0 = (PX)_0 = X_0$, and $\epsilon_0^X$ is the identity, so that $f_0 = g_0$ is the only solution.

Suppose now that $f$ has been defined up to the dimension $k$, satisfying the required properties, and consider the following diagram:

$$\begin{tikzcd}
S_{k+1} \arrow[rrrr, dotted, "f_{k+1}"] \arrow[ddddd, "j^S_{k+1}"'] \arrow[ddr, shift left] \arrow[ddr, shift right] & & & & (PX)_{k+1} \arrow[ddddd, "j^{PX}_{k+1}"] \arrow[ddl, shift left] \arrow[ddl, shift right] \\
& S_k \arrow[rr, "f_k"] \arrow[d, "j_k^S"] & & (PX)_k \arrow[d, "j_k^{PX}"'] & \\
& (QS)_k \arrow[rr, "(Qf)_k"] \arrow[dr, "g_k"'] & & (QPX)_k \arrow[dl, "\epsilon_k^X"] & \\
& & X_k & & \\
& & X_{k+1} \arrow[u, shift left] \arrow[u, shift right] & & \\
(QS)_{k+1} \arrow[uuur, shift left] \arrow[uuur, shift right] \arrow[urr, "g_{k+1}"] \arrow[rrrr, dotted, "(Qf)_{k+1}"'] & & & & (QPX)_{k+1} \arrow[uuul, shift left] \arrow[uuul, shift right] \arrow[ull, "\epsilon_{k+1}^X"']
\end{tikzcd}$$

Dotted arrows have not been defined yet but the remaining part is commutative. Now each $f_{k+1}$ such that the upper quadrangle $(S_{k+1}, (QS)_k, (QPX)_k, (PX)_{k+1})$ commutes extends $(f_0, \dots, f_k)$ to a morphism of $(k+1)$-polygraphs, hence determines a unique $(Qf)_{k+1}$

making the outer square commute. Now diagram chasing shows that there is a unique choice for which the bottom triangle also commutes, namely:

$$f_{k+1}u^{k+1} = (g_{k+1}j^S_{k+1}u^{k+1}, (Qf)_k su^{k+1}, (Qf)_k tu^{k+1}) \tag{9}$$

where $u^{k+1} \in S_{k+1}$.

■

4.5. Lifting. A key property of resolutions is that they lift morphisms. We begin with the following technical lemma.

4.6. Lemma. *Let $(S, \phi)$ be a resolution of $X$ and $k \geq 0$. Let $c^{k+1} \in X_{k+1}$ and $x^k_0$, $x^k_1$ two parallel cells of $S^*_k$ such that $\phi_k x^k_0 = c^k_0$ and $\phi_k x^k_1 = c^k_1$. There exists $z^{k+1} \in S^*_{k+1}$ such that $\phi_{k+1}z^{k+1} = c^{k+1}$, $z^k_0 \parallel x^k_0$ and $z^k_1 \parallel x^k_1$.*

Proof. Let $c^{k+1}$ as in the hypotheses of the lemma and consider for each integer $l$, $0 \leq l \leq k$, the property

**($\mathbf{P}_l$)** For each pair $x^l_0$, $x^l_1$ of parallel $l$-cells of $S^*_l$ such that $\phi_l x^l_e = c^l_e$, $e = 0, 1$, there exists $z^{k+1}$ such that $\phi_{k+1}z^{k+1} = c^{k+1}$ and $z^l_e \parallel c^l_e$.

Let us prove $\mathrm{P}_l$ by induction on $l \leq k$. $\mathrm{P}_0$ holds: take any antecedent of $c^{k+1}$ by $\phi_{k+1}$ (surjectivity), and recall that all 0-cells are parallel to each other.

Suppose now that $\mathrm{P}_l$ holds for $l < k$, and let $x^{l+1}_0 \parallel x^{l+1}_1$ with $\phi_l x^{l+1}_e = c^{l+1}_e$. By defining $x^l_0 = s_l x^{l+1}_e$ and $x^l_1 = t_l x^{l+1}_e$, we get $x^l_0 \parallel x^l_1$, and $\phi_l x^l_e = c^l_e$ because $\phi$ is a morphism.

By the induction hypothesis, we may chose $z^{k+1}$ above $c^{k+1}$ in such a way that $z^l_e \parallel x^l_e$. Point 2 in Definition 4.1 gives $a^{l+1} : x^l_0 \to z^l_0$ and $b^{l+1} : z^l_1 \to x^l_1$ such that

$$\begin{aligned}\phi_{l+1}a^{l+1} &= \mathrm{id}^{l+1}(\phi_l x^l_0) = \mathrm{id}^{l+1}(\phi_l z^l_0) = \mathrm{id}^{l+1}(c^l_0)\\ \phi_{l+1}b^{l+1} &= \mathrm{id}^{l+1}(\phi_l x^l_1) = \mathrm{id}^{l+1}(\phi_l z^l_1) = \mathrm{id}^{l+1}(c^l_1)\end{aligned}$$

Let

$$w^{k+1} = \mathrm{id}^{k+1}(a^{l+1}) *_l z^{k+1} *_l \mathrm{id}^{k+1}(a^{l+1})$$

We first get

$$\phi_{k+1}w^{k+1} = \phi_{k+1}z^{k+1} = c^{k+1}$$

on the other hand

$$\begin{aligned}w^{l+1}_0 &= a^{l+1} *_l z^{l+1}_0 *_l b^{l+1}\\ w^{l+1}_1 &= a^{l+1} *_l z^{l+1}_1 *_l b^{l+1}\end{aligned}$$

so that, for $e = 0, 1$,

$$\begin{aligned}s_l w^{l+1}_e &= s_l a^{l+1} = x^l_0 = s_l x^{l+1}_e\\ t_l w^{l+1}_e &= t_l b^{l+1} = x^l_1 = t_l x^{l+1}_e\end{aligned}$$

and $w^{l+1}_e \parallel x^{l+1}_e$. Whence $w^{k+1}$ satisfies the conditions of $\mathrm{P}_{l+1}$.

Thus $\mathrm{P}_l$ holds for all integers $l \leq k$, and especially for $k$ itself, but $\mathrm{P}_k$ is precisely the claim of the lemma. ■

As a corollary of Lemma 4.6, resolutions have a strong lifting property: if $x_0^k \parallel x_1^k$ and $\phi_k x_0^k \sim \phi_k x_1^k$, then $x_0^k \sim x_1^k$. It is enough to prove that $\phi_k x_0^k R_k \phi_k x_1^k$ implies $x_0^k \sim x_1^k$. Lemma 4.6 gives $z^{k+1}$ such that $\phi_{k+1} z^{k+1} = c^{k+1}$ and $z_e^k \parallel x_e^k$. On the other hand $\phi_k x_e^k = c_e^k = \phi_k z_e^k$, whence $a^{k+1} : x_0^k \to z_0^k$ and $b^{k+1} : z_1^k \to x_1^k$ by the stretching property. Thus

$$a^{k+1} *_k z^{k+1} *_k b^{k+1} : x_0^k \to x_1^k$$

and $x_0^k \sim x_1^k$.

It is now possible to prove the desired lifting property.

4.7. Proposition. *Let $X$ be an $\infty$-category, $S$ and $T$ polygraphs, $\phi : QS \to X$ a morphism and $\psi : QT \to X$ a resolution. Then there is a morphism $u : QS \to QT$ such that $\psi u = \phi$ (Figure 7).*

$$\begin{tikzcd} QS \arrow[r, "u"] \arrow[dr, "\phi"'] & QT \arrow[d, "\psi"] \\ & X \end{tikzcd}$$

Figure 7: lifting

Proof. We build $u_0, u_1, \ldots$ by induction on the dimension.

We first choose $u_0 : S_0^* \to T_0^*$ such that $\psi_0 u_0 = \phi_0$. This is possible because of the surjectivity of $\psi_0$.

Suppose now that $u$ has been defined up to dimension $k$, with

$$\psi_i u_i = \phi_i \qquad \text{for} \qquad 0 \leq i \leq k$$

We want a map $u_{n+1} : S_{n+1}^* \to T_{n+1}^*$ extending the given data to a morphism in $\mathbf{Cat}_{n+1}$. By the universal property of polygraphs, it suffices to define $u_{n+1}$ on the set $S_{n+1}$ of generators. Let then $x^{k+1} \in S_{k+1}$. As $x_0^k \parallel x_1^k$, we also have $u_k x_0^k \parallel u_k x_1^k$ and Lemma 4.6 yields $z^{k+1} \in T_{k+1}^*$ such that

$$\psi_{k+1} z^{k+1} \quad = \quad \phi_{k+1} x^{k+1} \tag{10}$$

and

$$z_e^k \parallel u_k x_e^k \quad \text{for} \quad e = 0, 1 \tag{11}$$

By successively applying $s_k$ and $t_k$ to the members of (10), we get:

$$\begin{aligned} \psi_k u_k x_0^k &= \phi_k x_0^k = s_k \phi_{k+1} x^{k+1} = s_k \psi_{k+1} z^{k+1} = \psi_k z_0^k \\ \psi_k u_k x_1^k &= \phi_k x_1^k = t_k \phi_{k+1} x^{k+1} = t_k \psi_{k+1} z^{k+1} = \psi_k z_1^k \end{aligned}$$

and, by (11), there are cells $a^{k+1} : u_k x_0^k \to z_0^k$ and $b^{k+1} : z_1^k \to x_1^k$ such that

$$\begin{array}{rcl}\psi_{k+1} a^{k+1} & = & \mathrm{id}^{k+1}(\psi_k u_k x_0^k) = \mathrm{id}^{k+1}(\psi_k z_0^k) \\ \psi_{k+1} b^{k+1} & = & \mathrm{id}^{k+1}(\psi_k u_k x_1^k) = \mathrm{id}^{k+1}(\psi_k z_1^k)\end{array}$$

It is now possible to define

$$u_{k+1} x^{k+1} = a^{k+1} *_k z^{k+1} *_k b^{k+1}$$

Thus

$$\begin{array}{rcl}s_k u_{k+1} x^{k+1} & = & u_k x_0^k \\ t_k u_{k+1} x^{k+1} & = & u_k x_1^k\end{array}$$

Hence $u_{k+1}$ extends $u$ to a morphism up to dimension $k+1$. Moreover

$$\begin{array}{rcl}\psi_{k+1} u_{k+1} x^{k+1} & = & (\psi_{k+1} a^{k+1}) *_k (\psi_{k+1} z^{k+1}) *_k (\psi_{k+1} b^{k+1}) \\ & = & \mathrm{id}^{k+1}(\psi_k u_k x_0^k) *_k (\phi_{k+1} x^{k+1}) *_k \mathrm{id}^{k+1}(\psi_k u_k x_1^k) \\ & = & \phi_{k+1} x^{k+1}\end{array}$$

which proves the property in dimension $k+1$. ∎

## 5. Homotopy theorem

In Proposition 4.7, the lifting morphism $u$ is of course not unique. Two such morphisms are however "homotopic", which of course needs a precise definition. For doing this, we associate to each $\infty$-category $X$ a new $\infty$-category $HX$, consisting very roughly of higher-dimensional paths in $X$. The details of the construction are found in appendix A. For the moment, the reader should look at Figure 11, as well as to the formulas (26) and (27), which give the source and target of the cells involved.

5.1. Theorem. *Let $X$ be an $\infty$-category, $S$ and $T$ polygraphs, $\phi : QS \to X$ a morphism and $\psi : QT \to X$ a resolution. If $u$, $v$ are morphisms $QS \to QT$ such that $\psi u = \psi v = \phi$, then there is a morphism $h : QS \to HQT$ such that $u = \mathrm{a}_+ h$ and $v = \mathrm{a}_- h$ (Figure 8).*

Proof. We build, in each dimension $i$, a map $h_i : (QS)_i \to (HQT)_i$ such that $h$ becomes a morphism $QS \to HQT$ and, for each $i$, $\mathrm{a}_+^i h_i = u_i$ and $\mathrm{a}_-^i h_i = v_i$. We proceed by induction on the dimension.

Let $x^0 \in (QS)_0 = S_0$. $u_0 x^0 \parallel v_0 x^0$ and $\psi_0 u_0 x^0 = \psi_0 v_0 x^0 = \phi_0 x^0$ by hypothesis. $(T, \psi)$ being a resolution, there is a 1-cell $w^1 \in (QT)_1$ such that

$$w^1 : u_0 x^0 \to v_0 x^0$$

$$\begin{tikzcd}
 & HQT \arrow[d, shift right, "\mathrm{a}_+"'] \arrow[d, shift left, "\mathrm{a}_-"] \\
QS \arrow[ur, "h"] \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[dr, "\phi"'] & QT \arrow[d, "\psi"] \\
 & X
\end{tikzcd}$$

Figure 8: homotopy

and

$$\psi_1 w^1 = \mathrm{id}^1(\psi_0 u_0 x^0)$$

By definition of $HQT$, $w^1 \in (HQT)_0$ and $[w^1] = w^1 \in (QT)_1$. We may define

$$h_0 x^0 = w^1$$

Thus

$$\begin{aligned} \mathrm{a}_+^0\, h_0 x^0 &= \mathrm{d}_-[w^1] = u_0 x^0 \\ \mathrm{a}_-^0\, h_0 x^0 &= \mathrm{d}_+[w^1] = v_0 x^0 \end{aligned}$$

and we are done in dimension 0.

Suppose now that we have defined, for each $0 \leq i \leq n$, a map

$$h_i : (QS)_i \to (HQT)_i$$

in such a way that $(h_0, \dots, h_n)$ is a morphism of $n$-categories, and for each $0 \leq i \leq n$,

$$\mathrm{a}_+^i\, h_i = u_i \tag{12}$$

$$\mathrm{a}_-^i\, h_i = v_i \tag{13}$$

and for each $x^i \in (QS)_i$,

$$\psi_{i+1}[h_i x^i] = \mathrm{id}^{i+1}(\psi_i u_i x^i) \tag{14}$$

We now define $h_{n+1} : (QS)_{n+1} \to (HQT)_{n+1}$ extending the previous data to a morphism of $(n+1)$-category, satisfying (12), (13) and (14) up to $i = n+1$. We first define a map

$$k_{n+1} : S_{n+1} \to (QHT)_{n+1}$$

Let $x^{n+1} \in S_{n+1}$ and consider the following expressions:

$$E = u_{n+1} x^{n+1} *_0 [h_0 x_+^0] *_1 \dots *_n [h_n x_+^n] \tag{15}$$

$$F = [h_n x_-^n] *_n \dots *_1 [h_0 x_-^0] *_0 v_{n+1} x^{n+1} \tag{16}$$

By induction hypothesis, $E$ and $F$ denote parallel cells in $(QT)_{n+1}$, and

$$\psi_{n+1}E = \psi_{n+1}u_{n+1}x^{n+1} = \phi_{n+1}x^{n+1}$$

$$\psi_{n+1}F = \psi_{n+1}v_{n+1}x^{n+1} = \phi_{n+1}x^{n+1}$$

so that there is a cell $w^{n+2} \in (QT)_{n+2}$ with source $E$, target $F$, and

$$\psi_{n+2}w^{n+2} = \mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}x^{n+1})$$

We may then define

$$k_{n+1}x^{n+1} = (h_n x^n_+, h_n x^n_-, u_{n+1}x^{n+1}, v_{n+1}x^{n+1}, w^{n+2}) \tag{17}$$

But $E$ and $F$ are easily seen to be $\mathrm{T}^{n+1}_- \, k_{n+1}x^{n+1}$ and $\mathrm{T}^{n+1}_+ \, k_{n+1}x^{n+1}$ respectively, so that $k_{n+1}x^{n+1}$ belongs to $(HQT)_{n+1}$. Note that

$$\mathrm{a}^{n+1}_+ \, k_{n+1} \quad = \quad u_{n+1} \tag{18}$$
$$\mathrm{a}^{n+1}_- \, k_{n+1} \quad = \quad v_{n+1} \tag{19}$$

and

$$\psi_{n+2}[k_{n+1}x^{n+1}] = \mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}x^{n+1}) \tag{20}$$

Moreover, $k_{n+1}$ extends the $h_i$'s to a morphism in $\mathbf{Cat}^+_n$. By the definition of polygraphs, there is a unique $h_{n+1} : (QS)_{n+1} \to (HQT)_{n+1}$ making $(h_i)_{0\leq i\leq n+1}$ a morphism in $\mathbf{Cat}_{n+1}$ and such that the following diagram commutes:

$$\begin{tikzcd} S_{n+1} \arrow[r, "j_{n+1}"] \arrow[dr, "k_{n+1}"'] & (QS)_{n+1} \arrow[d, "h_{n+1}"] \\ & (HQT)_{n+1} \end{tikzcd}$$

It remains to check that (18) and (19) extend to $h_{n+1}$, and that (14) still holds for $i = n+1$.

As for (18) and (19), just remember that every cell in $(QS)_{n+1}$ is a composite of elements of $S_{n+1}$ and identities on cells in $(QS)_n$, and that morphisms preserve composition and identities. Thus

$$\mathrm{a}^{n+1}_+ \, h_{n+1} \quad = \quad u_{n+1} \tag{21}$$
$$\mathrm{a}^{n+1}_- \, h_{n+1} \quad = \quad v_{n+1} \tag{22}$$

Let us show that

$$\psi_{n+2}[h_{n+1}x^{n+1}] = \mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}x^{n+1}) \tag{23}$$

for each $x^{n+1} \in (QS)_{n+1}$.

If $x^{n+1} \in j_{n+1}S_{n+1}$, (23) is exactly (20) and we are done.

If $x^{n+1} = \mathrm{I}x^n$ for some $x^n \in (QS)_n$,

$$\begin{aligned}
\psi_{n+2}[h_{n+1}x^{n+1}] &= \psi_{n+2}[h_{n+1}\mathrm{I}x^n] \\
&= \psi_{n+2}[\mathrm{I}h_n x^n] \\
&= \psi_{n+2}\mathrm{I}[h_n x^n] \\
&= \mathrm{I}\psi_{n+1}[h_n x^n] \\
&= \mathrm{I}\,\mathrm{id}^{n+1}(\psi_n u_n x^n) \\
&= \mathrm{id}^{n+2}(\mathrm{I}\psi_n u_n x^n) \\
&= \mathrm{id}^{n+2}(\psi_{n+1} u_{n+1} x^{n+1})
\end{aligned}$$

Suppose finally that (23) holds for $i$-composable cells $x^{n+1}$ and $y^{n+1}$, and check that it still holds for $z^{n+1} = x^{n+1} *_i y^{n+1}$. Let $k = n+1-i$.

$$\begin{aligned}
\psi_{n+2}[h_{n+1}z^{n+1}] &= \psi_{n+2}[h_{n+1}x^{n+1} *_i h_{n+1}y^{n+1}] \\
&= (\psi_{n+2}\,\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}h_{n+1}x^{n+1} *_i \psi_{n+2}[h_{n+1}y^{n+1}]) *_{i+1} \\
&\quad (\psi_{n+2}[h_{n+1}x^{n+1}] *_i \psi_{n+2}\,\mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}h_{n+1}y^{n+1}) \\
&= (\psi_{n+2}\,\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}h_{n+1}x^{n+1} *_i \mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}y^{n+1})) *_{i+1} \\
&\quad (\mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}x^{n+1}) *_i \psi_{n+2}\,\mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}h_{n+1}y^{n+1})
\end{aligned}$$

But

$$\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}h_{n+1}x^{n+1} = \mathrm{S}_-^{i+1}\,h_{i+1}x_-^{i+1}$$

so that

$$\begin{aligned}
\psi_{n+2}\,\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}h_{n+1}x^{n+1} &= \psi_{n+2}\mathrm{I}^k\,\mathrm{a}_+^{i+1}\,h_{i+1}x_-^{i+1} \\
&= \mathrm{I}^k\psi_{i+1}\,\mathrm{a}_+^{i+1}\,h_{i+1}x_-^{i+1} \\
&= \mathrm{I}^k\psi_{i+1}u_{i+1}x_-^{i+1}
\end{aligned}$$

by using the expression of $\mathrm{S}_-^{i+1}$, (14) and (21).

On the other hand

$$\mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}x^{n+1}) = \mathrm{I}\psi_{n+1}u_{n+1}x^{n+1}$$

and

$$\mathrm{d}_-^k\mathrm{I}\psi_{n+1}u_{n+1}x^{n+1} = \psi_{i+1}u_{i+1}x_-^{i+1}$$

The same argument shows that

$$\psi_{n+2}\,\mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}h_{n+1}y^{n+1} = \mathrm{I}^k\psi_{i+1}u_{i+1}y_+^{i+1}$$

and

$$\mathrm{d}_+^k\,\mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}y^{n+1}) = \psi_{i+1}u_{i+1}y_+^{i+1}$$

By applying exchange, we get

$$\begin{aligned}
\psi_{n+2}[h_{n+1}z^{n+1}] &= \mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}x^{n+1}) *_i \mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}y^{n+1}) \\
&= \mathrm{id}^{n+2}(\psi_{n+1}u_{n+1}z^{n+1})
\end{aligned}$$

■

## 6. Homological invariance

This section is devoted to the proof of the following corollary of Theorem 5.1.

6.1. Theorem. *Let $X$ be an $\infty$-category, and $(S, \phi)$, $(T, \psi)$ resolutions of $X$. The complexes $ZS$ and $ZT$ have the same homology.*

Thus $H_*(X)$ can be defined as $H_*(ZS)$, where $S$ is any polygraphic resolution of $X$.

Proof. Let $\phi : QS \to X$ a morphism, and $(T, \psi)$ a resolution of $X$, and $u$, $v$ two morphisms $QS \to QT$ with $\psi u = \psi v = \phi$. By Theorem 5.1, there is an $h : QS \to HQT$ such that $u = \mathrm{a}_+ \, h$ and $v = \mathrm{a}_- \, h$. The key point is that, when linearizing these equations, one gets an algebraic homotopy between $\tilde{u}$ and $\tilde{v}$ (see 3.3).

We will define, for each $i \geq 0$, a $\mathbf{Z}$-linear map $\theta_{i+1} : ZS_i \to ZT_{i+1}$ such that

$$\tilde{u}_i - \tilde{v}_i = \partial\theta_{i+1} + \theta_i\partial \tag{24}$$

where $\theta_0 = 0$ by convention (see Figure 9).

$$\begin{tikzcd}
ZS_{i-1} \arrow[dd, shift right, "\tilde{u}_{i-1}"'] \arrow[dd, shift left, "\tilde{v}_{i-1}"] \arrow[ddr, "\theta_i"] & ZS_i \arrow[l, "\partial"'] \arrow[dd, shift right, "\tilde{u}_i"'] \arrow[dd, shift left, "\tilde{v}_i"] \arrow[ddr, "\theta_{i+1}"] & ZS_{i+1} \arrow[l, "\partial"'] \arrow[dd, shift right, "\tilde{u}_{i+1}"'] \arrow[dd, shift left, "\tilde{v}_{i+1}"] \\
& & \\
ZT_{i-1} & ZT_i \arrow[l, "\partial"] & ZT_{i+1} \arrow[l, "\partial"]
\end{tikzcd}$$

Figure 9: algebraic homotopy

Now for each $i \geq 0$, we have a map $[\;] : (HQT)_i \to (QT)_{i+1}$ (see appendix A) so that we may also define a map

$$x^i \mapsto \lambda_{i+1}[h_i x^i]$$

from $S_i$ to $ZT_{i+1}$, which extends by linearity to $\theta_{i+1} : ZS_i \to ZT_{i+1}$. Let us assume for the moment that $\theta_{i+1}$ just defined satisfies

$$\theta_{i+1}\lambda_i x^i = \lambda_{i+1}[h_i x^i] \tag{25}$$

for all $x^i \in (QS)_i$ (see Figure 10).

We will return to (25) in Lemma 6.2 below.

Let us now evaluate $\partial\theta_{i+1}x^i$ for $x^i \in S_i$. First

$$\begin{aligned}
\partial\theta_{i+1}x^i &= \partial\lambda_{i+1}[h_i x^i] \\
&= \lambda_i t_i[h_i x^i] - \lambda_i s_i[h_i x^i]
\end{aligned}$$

$$\begin{tikzcd}
S_i \arrow[d] & (HQT)_i \arrow[d, "{[\,]}"] \\
(QS)_i \arrow[ur, "h_i"] \arrow[d, "\lambda_i"'] & (QT)_{i+1} \arrow[d, "\lambda_{i+1}"] \\
ZS_i \arrow[r, "\theta_{i+1}"'] & ZT_{i+1}
\end{tikzcd}$$

Figure 10: commutation of $\theta$

by using (5). But from the construction of $H$,

$$\begin{aligned} t_i[h_i x^i] &= \mathrm{T}^i_+\, h_i x^i \\ &= [\mathrm{d}_- h_i x^i] *_{i-1} \cdots *_1 [\mathrm{d}^i_- h_i x^i] *_0 \mathrm{a}^i_-\, h_i x^i \end{aligned}$$

In the last expression, only the two terms $[\mathrm{d}_- h_i x^i]$ and $\mathrm{a}^i_-\, h_i x^i$ are non-degenerate, the other ones are identities on cells of dimension $< i$. As linearization kills degenerate cells, we get

$$\begin{aligned} \lambda_i t_i[h_i x^i] &= \lambda_i[\mathrm{d}_- h_i x^i] + \lambda_i\, \mathrm{a}^i_-\, h_i x^i \\ &= \lambda_i[\mathrm{d}_- h_i x^i] + \lambda_i u_i x^i \\ &= \lambda_i[\mathrm{d}_- h_i x^i] + \tilde{u}_i x^i \end{aligned}$$

Likewise

$$\lambda_i s_i[h_i x^i] = \tilde{v}_i x^i + \lambda_i[\mathrm{d}_+ h_i x^i]$$

Thus

$$\partial\theta_{i+1} x^i = \tilde{u}_i x^i - \tilde{v}_i x^i + A$$

where

$$A = \lambda_i[\mathrm{d}_- h_i x^i] - \lambda_i[\mathrm{d}_+ h_i x^i]$$

Now, by using the fact that $h$ commutes with $\mathrm{d}_+$ and $\mathrm{d}_-$, together with (25) and the linearity of $\theta_i$, we get

$$\begin{aligned} A &= \lambda_i[\mathrm{d}_- h_i x^i] - \lambda_i[\mathrm{d}_+ h_i x^i] \\ &= \lambda_i[h_{i-1}\mathrm{d}_- x^i] - \lambda_i[h_{i-1}\mathrm{d}_+ x^i] \\ &= \theta_i\lambda_{i-1}\mathrm{d}_- x^i - \theta_i\lambda_{i-1}\mathrm{d}_+ x^i \\ &= \theta_i(\lambda_{i-1}\mathrm{d}_- x^i - \lambda_{i-1}\mathrm{d}_+ x^i) \\ &= -\theta_i\partial\lambda_i x^i \end{aligned}$$

and (24) follows, first for $x^i \in S_i$, then for any $x^i \in ZS_i$ by linearity.

In the case where $(S, \phi)$ and $(T, \psi)$ are both resolutions of $X$ we conclude by familiar arguments that $\tilde{u}$ induces an isomorphism on homology. ∎

It remains to check the small but crucial point of the equation (25). In the hypotheses of Theorem 6.1, we establish the following lemma:

6.2. LEMMA. *For each $i \geq 0$, there is a $\mathbf{Z}$-linear map $\theta_{i+1} : ZS_i \to ZT_{i+1}$ such that (25), that is, for each $x^i \in (QS)_i$,*

$$\theta_{i+1}\lambda_i x^i = \lambda_{i+1}[h_i x^i]$$

PROOF. Define $\theta_{i+1}$ as in the proof of Theorem 6.1. Then we show (25) by induction on the complexity of $x^i$.

- If $x^i \in S_i$, this is just the definition of $\theta_{i+1}$;
- if $x^i = \mathrm{id}^i(x^j)$, where $j < i$, $\lambda_i x^i = 0$, and the left member of (25) is zero by linearity of $\theta_{i+1}$. On the other hand, $h$ is a morphism, hence $h_i x^i = \mathrm{id}^i(h_j x^j)$, and by (44) $[h_i x^i] = \mathrm{id}^{i+1}([h_j x^j])$, so that $\lambda_{i+1}[h_i x^i] = 0$ and we are done in this case;
- if $x^i = y^i *_j z^i$ for smaller cells $y^i$ and $z^i$, and $j < i$, the induction hypothesis gives

$$\begin{aligned}\theta_{i+1}\lambda_i x^i &= \theta_{i+1}\lambda_i y^i + \theta_{i+1}\lambda_i z^i \\ &= \lambda_{i+1}[h_i y^i] + \lambda_{i+1}[h_i z^i]\end{aligned}$$

  On the other hand, as $h$ is a morphism

$$\lambda_{i+1}[h_i x^i] = \lambda_{i+1}[h_i y^i *_j h_i z^i]$$

  Now the equation (47) of appendix A shows how [ ] behaves with respect to composition. In particular, only terms within brackets are non-degenerated, the other ones being killed by linearization, whence

$$\lambda_{i+1}[h_i y^i *_j h_i z^i] = \lambda_{i+1}[h_i y^i] + \lambda_{i+1}[h_i z^i]$$

  and we are done again.

Thus the lemma is proved. ∎

Let us point out that (25) is the only place of our argument where we really need $h$ as a morphism between $\infty$-categories, not just between $\infty$-graphs.

## 7. Conclusion

As we indicated before, works on finite derivation types are a main source of the present notion of resolution. Precisely, if we consider a monoid $X$ as a particular case of $\infty$-category, it has finite derivation type if and only if it has a resolution $(S, \phi)$ with finite $S_3$. This immediately suggests a notion of finite derivation type in any dimension, but we need invariance properties of this notion, presumably based on Theorem 5.1.

Another remark is that our resolutions are still too big for practical computations Therefore, generalized versions of Squier's theorems should be established, and will be the subject of further work.

Let us simply point out for the moment that the relationship between finiteness and confluence becomes much more intricate in dimensions $\geq 2$ than in the case of string rewriting, so that even asking the correct questions seems far from obvious.

Finally, very similar motivations and techniques appear in recent works on concurrency, and we expect fruitful interactions in that direction (see [9, 10]).

**Acknowledgment**

I wish to thank Albert Burroni for his invaluable help.

## A. A category of paths

Given an $\infty$-category $X$, we define an $\infty$-graph $HX$ with identities and compositions and show how $HX$ becomes itself an $\infty$-category. We define sets $(HX)_n$ by induction, together with maps:

$$(HX)_{n-1} \overset{\mathrm{d}_+}{\underset{\mathrm{d}_-}{\leftleftarrows}} (HX)_n$$

$$(HX)_n \overset{\mathrm{a}^n_+}{\underset{\mathrm{a}^n_-}{\rightrightarrows}} X_n$$

$$(HX)_n \xrightarrow{[\ ]} X_{n+1}$$

Figure 11 gives an insight into $H$ in small dimensions. In particular the cylinder represents $[x^2]$, which has to be oriented from the front to the bottom (dotted lines). Similar pictures already appear in [2], and since then in many works (see for instance chapter 3 of [7]), though in a different context: we stress here the fact that the cells of $HX$ are built from material already present in $X$. Also the construction is carried out in arbitrary dimensions.

From the above symbols we may build the following formal expressions which play a crucial role in the construction:

$$\begin{aligned} \mathrm{T}^0_-\, x &= \mathrm{a}^0_+\, x \\ \mathrm{T}^0_+\, x &= \mathrm{a}^0_-\, x \\ \mathrm{T}^1_-\, x &= \mathrm{a}^1_+\, x *_0 [\mathrm{d}_+ x] \\ \mathrm{T}^1_+\, x &= [\mathrm{d}_- x] *_0 \mathrm{a}^1_-\, x \end{aligned}$$

and for $n \geq 2$

$$\mathrm{T}^n_-\, x = \mathrm{a}^n_+\, x *_0 [\mathrm{d}^n_+ x] *_1 [\mathrm{d}^{n-1}_+ x] *_2 \cdots *_{n-1} [\mathrm{d}_+ x] \tag{26}$$

$$\mathrm{T}^n_+\, x = [\mathrm{d}_- x] *_{n-1} \cdots *_2 [\mathrm{d}^{n-1}_- x] *_1 [\mathrm{d}^n_- x] *_0 \mathrm{a}^n_-\, x \tag{27}$$

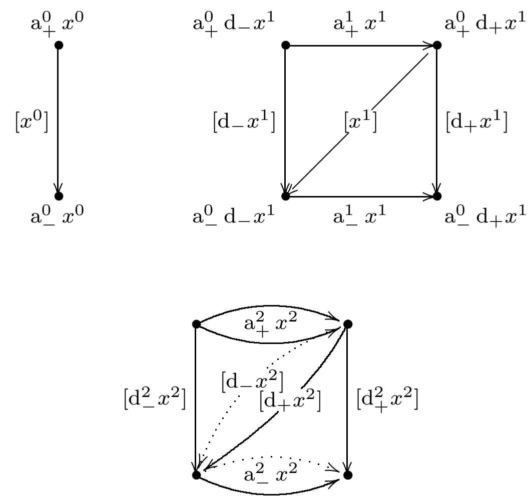


Figure 11: $HX$ in dimension 0, 1, 2

Why the signs on the right hand side of (26) and (27) do no agree with the one of the left hand side will be explained later. Eventually $\mathrm{T}_-^k$ and $\mathrm{T}_+^k$ become maps $(HX)_k \to X_k$. Also the following truncated expressions will be useful for technical purposes:

$$\mathrm{T}_-^{n,l}\, x \;=\; \mathrm{a}_+^n\, x *_0 [\mathrm{d}_+^n x] *_1 [\mathrm{d}_+^{n-1} x] *_2 \cdots *_{n-l} [\mathrm{d}_+^l x] \tag{28}$$
$$\mathrm{T}_+^{n,l}\, x \;=\; [\mathrm{d}_-^l x] *_{n-l} \cdots *_2 [\mathrm{d}_-^{n-1} x] *_1 [\mathrm{d}_-^n x] *_0 \mathrm{a}_-^n\, x \tag{29}$$

for each $l \in \{1, \ldots, n+1\}$. As the case is of particular importance we define $\mathrm{S}_-^n\, x = \mathrm{T}_-^{n,2}\, x$ and $\mathrm{S}_+^n\, x = \mathrm{T}_+^{n,2}\, x$.

For $n = 0$, $(HX)_0$ will be simply $X_1$, $[\ ]$ is the identity. When we write $[x]$ it is the cell of $X_1$ we have in mind. For instance $\mathrm{a}_+^0$ and $\mathrm{a}_-^0$ are defined by:

$$\mathrm{a}_+^0\, x \;=\; \mathrm{d}_-[x] \tag{30}$$
$$\mathrm{a}_-^0\, x \;=\; \mathrm{d}_+[x] \tag{31}$$

here $\mathrm{d}_-$ and $\mathrm{d}_+$ are of course the source and target on $X_1$. Notice that $\mathrm{T}_-^0$ and $\mathrm{T}_+^0$ define maps $(HX)_0 \to X_0$, because they are just $\mathrm{a}_+^0$ and $\mathrm{a}_-^0$ in this case.

A.1. Induction step. Suppose we have defined so far an $n$-category

$$(HX)_0 \underset{\mathrm{d}_-}{\overset{\mathrm{d}_+}{\leftleftarrows}} (HX)_1 \underset{\mathrm{d}_-}{\overset{\mathrm{d}_+}{\leftleftarrows}} \cdots \underset{\mathrm{d}_-}{\overset{\mathrm{d}_+}{\leftleftarrows}} (HX)_n$$

with maps $[\ ]$, $\mathrm{a}_+^k$, $\mathrm{a}_-^k$ satisfying the following conditions:

**(H1)** $\mathrm{a}_+$ and $\mathrm{a}_-$ define morphisms of $n$-graphs, that is for all signs $\epsilon$, $\epsilon'$ and each index $0 \leq i \leq n-1$ we get a commutative diagram

$$\begin{tikzcd}
(HX)_i \arrow[d, "\mathrm{a}^i_{\epsilon'}"'] & (HX)_{i+1} \arrow[l, "\mathrm{d}_\epsilon"'] \arrow[d, "\mathrm{a}^{i+1}_{\epsilon'}"] \\
X_i & X_{i+1} \arrow[l, "\mathrm{d}_\epsilon"]
\end{tikzcd}$$

**(H2)** For each $i \in \{0, \dots, n\}$ and $x \in (HX)_i$, the expressions $\mathrm{T}^i_-\, x$ and $\mathrm{T}^i_+\, x$ are well defined, they denote $i$-cells in $X$ and $[x]$ is an $(i+1)$-cell in $X$ such that

$$\mathrm{T}^i_-\, x \xrightarrow{[x]} \mathrm{T}^i_+\, x \tag{32}$$

We extend these data to an $(n+1)$-graph by defining a set $(HX)_{n+1}$ and morphisms

$$(HX)_n \underset{\mathrm{d}_-}{\overset{\mathrm{d}_+}{\leftleftarrows}} (HX)_{n+1}$$

as follows: a cell $x$ in $(HX)_{n+1}$ is a 5-tuple

$$(x^n_+, x^n_-, v^{n+1}_+, v^{n+1}_-, w^{n+2}) \tag{33}$$

whose components are subject to the following conditions:

**(C0)** $x^n_+$ and $x^n_-$ are parallel cells in $(HX)_n$.

**(C1)** $v^{n+1}_+$ and $v^{n+1}_-$ are $(n+1)$-cells in $X_n$ such that

$$\mathrm{a}^n_+\, x^n_- \xrightarrow{v^{n+1}_+} \mathrm{a}^n_+\, x^n_+$$

$$\mathrm{a}^n_-\, x^n_- 0 \xrightarrow{v^{n+1}_-} \mathrm{a}^n_-\, x^n_+$$

We define $\mathrm{d}_+ x = x^n_+$, $\mathrm{d}_- x = x^n_-$, $\mathrm{a}^{n+1}_+\, x = v^{n+1}_+$ and $\mathrm{a}^{n+1}_-\, x = v^{n+1}_-$. As a consequence condition (H1) still holds for $i = n$.

**(C2)** $w^{n+2}$ is an $(n+2)$-cell in $X_{n+2}$ such that

$$\mathrm{T}^{n+1}_-\, x \xrightarrow{w^{n+2}} \mathrm{T}^{n+1}_+\, x$$

and $[x] = w^{n+2}$ so that (32) still holds for $i = n+1$.

That we get an $(n+1)$-graph is obvious from the definition of $\mathrm{d}_+$ and $\mathrm{d}_-$, and the induction hypothesis. Notice that the above definitions do not guarantee that $(HX)_{n+1} \neq \emptyset$. Now we need to define $(n+1)$-dimensional identities on $i$-cells as well as $i$-dimensional composites of $(n+1)$-cells, for all $0 \leq i \leq n$, and check that these operations match the axioms of $(n+1)$-category.

A.2. Some properties of $\mathrm{T}_+$ and $\mathrm{T}_-$.

A.3. Lemma. *For all $x \in (HX)_{n+1}$*

$$\begin{aligned} \mathrm{d}_- \mathrm{T}_-^{n+1} x &= \mathrm{T}_-^n \mathrm{d}_- x \\ \mathrm{d}_+ \mathrm{T}_-^{n+1} x &= \mathrm{T}_+^n \mathrm{d}_+ x \\ \mathrm{d}_- \mathrm{T}_+^{n+1} x &= \mathrm{T}_-^n \mathrm{d}_- x \\ \mathrm{d}_+ \mathrm{T}_+^{n+1} x &= \mathrm{T}_+^n \mathrm{d}_+ x \end{aligned}$$

Proof. Let $n \geq 1$ and $x$ be in $(HX)_{n+1}$. Consider

$$\mathrm{T}_-^{n+1} x = \mathrm{a}_+^{n+1} x *_0 [\mathrm{d}_+^{n+1} x] *_1 \cdots *_n [\mathrm{d}_+ x] \tag{34}$$

The construction of $(HX)_{n+1}$ requires that this composite be defined as a cell in $X_{n+1}$. $\mathrm{a}_+^{n+1} x \in X_{n+1}$ and for each $i \in \{2, \ldots, n+1\}$, $[\mathrm{d}_+^i x]$ denotes here the $(n+1)$-dimensional identity on a cell in $X_{n+2-i}$, so that for $i \geq 2$

$$\mathrm{d}_-[\mathrm{d}_+^i x] = \mathrm{d}_+[\mathrm{d}_+^i x] = [\mathrm{d}_+^i x] \tag{35}$$

Thus

$$\begin{aligned} \mathrm{d}_- \mathrm{T}_-^{n+1} x &= \mathrm{d}_-(\mathrm{a}_+^{n+1} x *_0 [\mathrm{d}_+^{n+1} x] *_1 \cdots *_{n-1} [\mathrm{d}_+^2 x]) \\ &= \mathrm{d}_- \mathrm{a}_+^{n+1} x *_0 \mathrm{d}_-[\mathrm{d}_+^{n+1} x] *_1 \cdots *_{n-1} \mathrm{d}_-[\mathrm{d}_+^2 x] \\ &= \mathrm{a}_+^n \mathrm{d}_- x *_0 [\mathrm{d}_+^{n+1} x] *_1 \cdots *_{n-1} [\mathrm{d}_+^2 x] \\ &= \mathrm{a}_+^n \mathrm{d}_- x *_0 [\mathrm{d}_+^n \mathrm{d}_+ x] *_1 \cdots *_{n-1} [\mathrm{d}_+ \mathrm{d}_+ x] \\ &= \mathrm{T}_-^n \mathrm{d}_- x \end{aligned}$$

As regards the second equality

$$\mathrm{d}_+ \mathrm{T}_-^{n+1} x = \mathrm{d}_+[\mathrm{d}_+ x] = \mathrm{T}_+^n \mathrm{d}_+ x \tag{36}$$

from the definition of [ ] on $(HX)_n$.

By symmetry we get the two remaining equalities as well. ∎

Also, for all $0 < k < l$ and all signs $\epsilon$, $\epsilon'$, we get

$$\mathrm{d}_\epsilon^k \mathrm{T}_{\epsilon'}^{n,l} = \mathrm{T}_{\epsilon'}^{n-k,l-k} \mathrm{d}_\epsilon^k \tag{37}$$

as in the proof of Lemma A.3. The next lemma shows that our formal expression define actual cells:

A.4. Lemma. *Suppose we are given $x_+^n$, $x_-^n$ two parallel cells in $(HX)_n$, as well as $v_+^{n+1}$, $v_-^{n+1}$ satisfying* (C1). *Then*

$$v_+^{n+1} *_0 [\mathrm{d}_+^n x_+^n] *_1 [\mathrm{d}_+^{n-1} x_+^n] *_2 \cdots *_n [x_+^n] \tag{38}$$

*and*

$$[x_-^n] *_n \cdots *_2 [\mathrm{d}_-^{n-1} x_-^n] *_1 [\mathrm{d}_-^n x_-^n] *_0 v_-^{n+1} \tag{39}$$

*are well defined and denote $(n+1)$-cells in $X$.*

PROOF. Let us prove that (38) is well defined; define

$$u_i = v_+^{n+1} *_0 [\mathrm{d}_+^n x_+^n] *_1 [\mathrm{d}_+^{n-1} x_+^n] *_2 \cdots *_i [\mathrm{d}_+^{n-i} x_+^n] \tag{40}$$

we show by induction on $i$ that $u_i$ is well defined for $i \in \{0, \ldots, n\}$.

$u_0 = v_+^{n+1} *_0 [\mathrm{d}_+^n x_+^n]$ where $[\mathrm{d}_+^n x_+^n]$ is in fact a $(n+1)$-identity on a 1-cell. Thus

$$\mathrm{d}_-^{n+1}[\mathrm{d}_+^n x_+^n] = \mathrm{d}_-[\mathrm{d}_+^n x_+^n] \tag{41}$$

On the other hand

$$\mathrm{d}_+^{n+1} v_+^{n+1} = \mathrm{d}_+^n \mathrm{d}_+ v_+^{n+1} = \mathrm{d}_+^n\, \mathrm{a}_+^n\, x_+^n = \mathrm{a}_+^0\, \mathrm{d}_+^n x_+^n = \mathrm{d}_-[\mathrm{d}_+^n x_+^n] \tag{42}$$

using (C1) and (30). As a consequence, $v_+^{n+1}$ and $[\mathrm{d}_+^n x_+^n]$ are 0-composable and $u_0$ is well defined and belongs to $X_{n+1}$.

Suppose now that $u_i$ is well defined for an $i < n$. Because all factors but $v_+^{n+1}$ and $[x_+^n]$ are identities, we get

$$\begin{array}{rcl}
\mathrm{d}_+^{n-i} u_i & = & \mathrm{d}_+^{n-i} v_+^{n+1} *_0 [\mathrm{d}_+^n x_+^n] *_1 \cdots *_i [\mathrm{d}_+^{n-i} x_+^n] \\
& = & \mathrm{d}_+^{n-i-1}\, \mathrm{a}_+^n\, x_+^n *_0 [\mathrm{d}_+^n x_+^n] *_1 \cdots *_i [\mathrm{d}_+^{n-i} x_+^n] \\
& = & \mathrm{a}_+^{i+1}\, \mathrm{d}_+^{n-i-1} x_+^n *_0 [\mathrm{d}_+^n x_+^n] *_1 \cdots *_i [\mathrm{d}_+^{n-i} x_+^n] \\
& = & \mathrm{T}_-^{i+1}\, \mathrm{d}_+^{n-i-1} x_+^n
\end{array}$$

Now

$$\mathrm{d}_-^{n-i}[\mathrm{d}_+^{n-i-1} x_+^n] = \mathrm{d}_-[\mathrm{d}_+^{n-i-1} x_+^n] = \mathrm{T}_-^{i+1}\, \mathrm{d}_+^{n-i-1} x_+^n$$

It shows that $u_i$ is $(i+1)$-composable with $[\mathrm{d}_+^{n-i-1} x_+^n]$, and that $u_{i+1}$ is well defined. This gives the result for (38). (39) is proved accordingly. ∎

A.5. CONSTRUCTION OF IDENTITY CELLS. As a consequence, for each $x^n \in (HX)_n$, we may define a cell in $(HX)_{n+1}$ which eventually becomes the identity on $x^n$. Precisely:

$$x^{n+1} = (x^n, x^n, v_+^{n+1}, v_-^{n+1}, w^{n+2}) \tag{43}$$

where

$$\begin{array}{rcl}
v_+^{n+1} & = & \mathrm{id}^{n+1}(\mathrm{a}_+^n\, x^n) \\
v_-^{n+1} & = & \mathrm{id}^{n+1}(\mathrm{a}_-^n\, x^n)
\end{array}$$

and

$$w^{n+2} = \mathrm{id}^{n+2}([x^n]) \tag{44}$$

The first four components clearly satisfy (C0-1). On the other hand Lemma A.4 proves that the expressions

$$v_+^{n+1} *_0 [\mathrm{d}_+^n x^n] *_1 [\mathrm{d}_+^{n-1} x^n] *_2 \cdots *_n [x^n]$$

and

$$[x^n] *_n \cdots *_2 [\mathrm{d}_-^{n-1} x^n] *_1 [\mathrm{d}_-^n x^n] *_0 v_-^{n+1}$$

are well-defined. But they are exactly $\mathrm{T}_-^{n+1}\, x^{n+1}$ and $\mathrm{T}_+^{n+1}\, x^{n+1}$. Now a closer inspection of the formulas shows that in this particular case:

$$\begin{aligned} \mathrm{T}_-^{n+1}\, x^{n+1} &= \mathrm{id}^{n+1}(\mathrm{T}_-^n\, x^n) *_n [x^n] \\ \mathrm{T}_+^{n+1}\, x^{n+1} &= [x^n] *_n \mathrm{id}^{n+1}(\mathrm{T}_+^n\, x^n) \end{aligned}$$

but $\mathrm{d}_-[x^n] = \mathrm{T}_-^n\, x^n$ and $\mathrm{d}_+[x^n] = \mathrm{T}_+^n\, x^n$, so that

$$\mathrm{T}_-^{n+1}\, x^{n+1} = \mathrm{T}_-^{n+1}\, x^{n+1} = [x^n]$$

and

$$\mathrm{T}_-^{n+1}\, x^{n+1} \xrightarrow{w^{n+2}} \mathrm{T}_+^{n+1}\, x^{n+1}$$

Thus $(HX)_{n+1}$ contains at least as many cells as $(HX)_n$.

For readability, the previous construction will be denoted by the same symbol, say I, in each dimension; here for instance

$$x^{n+1} = \mathrm{I} x^n$$

furthermore, if $x^i \in (HX)_i$, and $k \geq 0$, $\mathrm{I}^k x^i$ will be the $(i+k)$-cell built from $x^i$ by $k$ successive applications of the construction. Thus the symbol I behaves very much like $\mathrm{d}_+$ and $\mathrm{d}_-$. Notice also that, for each $l \leq k$,

$$\mathrm{d}_+^l \mathrm{I}^k x = \mathrm{d}_-^l \mathrm{I}^k x = \mathrm{I}^{k-l} x$$

A.6. Composition of cells. Now we have to define the $i$-dimensional composition between cells in $(HX)_{n+1}$ in such a way that we get an $n+1$-category. Suppose this has been done up to $(HX)_n$. Suppose in addition that for each $0 \leq i < m \leq n$ and $i$-composable $m$-cells $u$ and $v$, the composite $w = u *_i v$ satisfies:

$$[w] = (\mathrm{S}_-^{i+1}\, \mathrm{d}_-^{l-1} u *_i [v]) *_{i+1} ([u] *_i \mathrm{S}_+^{i+1}\, \mathrm{d}_+^{l-1} u) \tag{45}$$

where $l = m - i$. Where the above equation comes from will be explained during the induction process. Take now $x$, $y$ two cells in $(HX)_{n+1}$. Let $k \in \{1, \ldots, n+1\}$ and $i = n+1-k$. $x$ and $y$ will be $i$-composable iff

$$\mathrm{d}_+^k x = \mathrm{d}_-^k y$$

Recall that

$$\begin{aligned} x &= (\mathrm{d}_+ x, \mathrm{d}_- x, \mathrm{a}_+^{n+1}\, x, \mathrm{a}_-^{n+1}\, x, [x]) \\ y &= (\mathrm{d}_+ y, \mathrm{d}_- y, \mathrm{a}_+^{n+1}\, y, \mathrm{a}_-^{n+1}\, y, [y]) \end{aligned}$$

We define $z = x *_i y$ component by component:

- If $k = 1$, $i = n$
$$\mathrm{d}_+ z = \mathrm{d}_+ y \qquad \mathrm{d}_- z = \mathrm{d}_- x$$
If $k > 1$, $\mathrm{d}_+^{k-1}\mathrm{d}_+ x = \mathrm{d}_+^k x = \mathrm{d}_-^k y = \mathrm{d}_-^{k-1}\mathrm{d}_+ y$ so that $\mathrm{d}_+ x$ and $\mathrm{d}_+ y$ are $i$-composable. Likewise $\mathrm{d}_- x$ and $\mathrm{d}_- y$ are $i$-composable and we may define
$$\mathrm{d}_+ z = \mathrm{d}_+ x *_i \mathrm{d}_+ y \qquad \mathrm{d}_- z = \mathrm{d}_- x *_i \mathrm{d}_- y$$

- By (H1)
$$\mathrm{d}_+^k\, \mathrm{a}_+^{n+1}\, x = \mathrm{a}_+^i\, \mathrm{d}_+^k x = \mathrm{a}_+^i\, \mathrm{d}_-^k y = \mathrm{d}_-^k\, \mathrm{a}_+^{n+1}\, y$$
hence $\mathrm{a}_+^{n+1}\, x$ and $\mathrm{a}_+^{n+1}\, y$ are $i$-composable, and the same holds for $\mathrm{a}_-^{n+1}\, x$ and $\mathrm{a}_-^{n+1}\, y$ such that we may define
$$\mathrm{a}_+^{n+1}\, z = \mathrm{a}_+^{n+1}\, x *_i \mathrm{a}_+^{n+1}\, y \qquad \mathrm{a}_-^{n+1}\, z = \mathrm{a}_-^{n+1}\, x *_i \mathrm{a}_-^{n+1}\, y \tag{46}$$

- Recall that $[x] : \mathrm{T}_-^{n+1}\, x \to \mathrm{T}_+^{n+1}\, x$. Lemma A.3 shows that
$$\begin{array}{rcl}
\mathrm{d}_+^k[x] & = & \mathrm{d}_+^{k-1}\mathrm{d}_+[x] \\
& = & \mathrm{d}_+^{k-1}\, \mathrm{T}_+^{n+1}\, x \\
& = & \mathrm{T}_+^{n-k+2}\, \mathrm{d}_+^{k-1} x \\
& = & [\mathrm{d}_-^k x] *_{n+1-k} \mathrm{S}_+^{n-k+2}\, \mathrm{d}_+^{k-1} x \\
& = & [\mathrm{d}_-^k x] *_i \mathrm{S}_+^{i+1}\, \mathrm{d}_+^{k-1} x
\end{array}$$
Likewise $\mathrm{d}_-^k[x] = \mathrm{S}_-^{i+1}\, \mathrm{d}_-^{k-1} *_i [\mathrm{d}_+^k x]$ and the same relations hold for $[y]$. But here $\mathrm{d}_+^k x = \mathrm{d}_-^k y$, so that $[x]$ and $[y]$ look like:
$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{S}_-^{i+1}\, \mathrm{d}_-^{k-1} x"] \arrow[d, "{[\mathrm{d}_-^k x]}"'] & \bullet \arrow[r, "\mathrm{S}_-^{i+1}\, \mathrm{d}_-^{k-1} y"] \arrow[d, "{[\mathrm{d}_+^k x]}"', "{[\mathrm{d}_-^k y]}"] \arrow[dl, "{[x]}"'] & \bullet \arrow[d, "{[\mathrm{d}_+^k y]}"] \arrow[dl, "{[y]}"'] \\
\bullet \arrow[r, "\mathrm{S}_+^{i+1}\, \mathrm{d}_+^{k-1} x"'] & \bullet \arrow[r, "\mathrm{S}_+^{i+1}\, \mathrm{d}_+^{k-1} y"'] & \bullet
\end{tikzcd}$$
the above diagram being drawn in the 2-category:
$$\begin{tikzcd}
X_i & X_{i+1} \arrow[l, shift right, "\mathrm{d}_+"'] \arrow[l, shift left, "\mathrm{d}_-"] & X_{n+2} \arrow[l, shift right, "\mathrm{d}_+^k"'] \arrow[l, shift left, "\mathrm{d}_-^k"]
\end{tikzcd}$$
It is then possible to define:
$$[z] = (\mathrm{S}_-^{i+1}\, \mathrm{d}_-^{k-1} x *_i [y]) *_{i+1} ([x] *_i \mathrm{S}_+^{i+1}\, \mathrm{d}_+^{k-1} y) \tag{47}$$

We are now able to prove the following

A.7. Proposition. *The* 5*-tuple*

$$(\mathrm{d}_+ z, \mathrm{d}_- z, \mathrm{a}_+^{n+1} z, \mathrm{a}_-^{n+1} z, [z])$$

*satisfies the conditions* (C0), (C1), *and* (C2).

Proof. This is straightforward for (C0) and (C1). As for (C2) we know that $[z] \in X_{n+2}$ and we must show that $\mathrm{d}_+[z] = \mathrm{T}_+^{n+1} z$ and $\mathrm{d}_-[z] = \mathrm{T}_-^{n+1} z$. We consider two cases (1) $k = 1$ and (2) $k > 1$.

$\boxed{\text{Case 1}}$. Suppose $k = 1$. Then $i = n$ and (47) becomes

$$[z] = (\mathrm{S}_-^{n+1} x *_n [y]) *_{n+1} ([x] *_n \mathrm{S}_+^{n+1} y)$$

hence

$$\begin{aligned}
\mathrm{d}_-[z] &= \mathrm{d}_-(\mathrm{S}_-^{n+1} x *_n [y]) \\
&= \mathrm{S}_-^{n+1} x *_n \mathrm{d}_-[y] \\
&= \mathrm{S}_-^{n+1} x *_n \mathrm{T}_-^{n+1} y \\
&= \mathrm{S}_-^{n+1} x *_n \mathrm{S}_-^{n+1} y *_n [\mathrm{d}_+ y]
\end{aligned}$$

But $[\mathrm{d}_+^l x] = [\mathrm{d}_+^l y]$ for each $l \in \{2, \ldots, n+1\}$ and they are all identities in $\mathrm{S}_-^{n+1} x$, $\mathrm{S}_-^{n+1} y$. Also $\mathrm{a}_-^{n+1} z = \mathrm{a}_-^{n+1} x *_n \mathrm{a}_-^{n+1} y$ and the exchange rule applies, giving

$$\mathrm{d}_-[z] = \mathrm{T}_-^{n+1} z$$

Likewise $\mathrm{d}_+[z] = \mathrm{T}_+^{n+1} z$ and we get the result.

$\boxed{\text{Case 2}}$. Suppose $k > 1$. Here

$$\mathrm{d}_-[z] = (\mathrm{S}_-^{i+1} \mathrm{d}_-^{k-1} x *_i \mathrm{T}_-^{n+1} y) *_{i+1} (\mathrm{T}_-^{n+1} x *_i \mathrm{S}_+^{i+1} \mathrm{d}_+^{k-1} y) \tag{48}$$

$$\mathrm{d}_+[z] = (\mathrm{S}_-^{i+1} \mathrm{d}_-^{k-1} x *_i \mathrm{T}_+^{n+1} y) *_{i+1} (\mathrm{T}_+^{n+1} x *_i \mathrm{S}_+^{i+1} \mathrm{d}_+^{k-1} y) \tag{49}$$

and we have to prove that these expressions are respectively $\mathrm{T}_-^{n+1} z$ and $\mathrm{T}_+^{n+1} z$. But this is the particular case $l = 1$ in the next lemma. ■

A.8. Lemma. *Let* $x$, $y$, $z$ *be as above, and* $k \geq 2$. *Then*

$$\begin{aligned}
(\mathrm{S}_-^{i+1} \mathrm{d}_-^{k-1} x *_i \mathrm{T}_-^{n+1,l} y) *_{i+1} (\mathrm{T}_-^{n+1,l} x *_i \mathrm{S}_+^{i+1} \mathrm{d}_+^{k-1} y) &= \mathrm{T}_-^{n+1,l} z \\
(\mathrm{S}_-^{i+1} \mathrm{d}_-^{k-1} x *_i \mathrm{T}_+^{n+1,l} y) *_{i+1} (\mathrm{T}_+^{n+1,l} x *_i \mathrm{S}_+^{i+1} \mathrm{d}_+^{k-1} y) &= \mathrm{T}_+^{n+1,l} z
\end{aligned}$$

*for all* $l \in 1, \ldots, k-1$.

Proof. By induction on $k-1-l$. Let us first examine the case $l = k-1$. We notice that

$$\mathrm{T}_-^{n+1,k+1}\,x *_i \mathrm{T}_-^{n+1,k+1}\,y = \mathrm{T}_-^{n+1,k+1}\,z \tag{50}$$

because $\mathrm{a}_+^{n+1}\,z = \mathrm{a}_+^{n+1}\,x *_i \mathrm{a}_+^{n+1}\,y$ and $[\mathrm{d}_+^m x] = [\mathrm{d}_+^m y] = [\mathrm{d}_+^m z]$ for all $m \geq k+1$, and are all identities in (50). Let

$$A = (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{T}_-^{n+1,k-1}\,y) *_{i+1} (\mathrm{T}_-^{n+1,k-1}\,x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y)$$

By using the exchange rule and the properties of the identities we get

$$\begin{aligned}
A &= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{T}_-^{n+1,k-1}\,y) *_{i+1} \\
&\quad (((\mathrm{T}_-^{n+1,k+1}\,x *_i [\mathrm{d}_+^k x]) *_{i+1} [\mathrm{d}_+^{k-1}x]) *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) \\
&= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{T}_-^{n+1,k-1}\,y) *_{i+1} (\mathrm{T}_-^{n+1,k+1}\,x *_i [\mathrm{d}_+^k x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) \\
&\quad *_{i+1}([\mathrm{d}_+^{k-1}x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) \\
&= (\mathrm{T}_-^{n+1,k+1}\,x *_i \mathrm{T}_-^{n+1,k-1}\,y) *_{i+1} ([\mathrm{d}_+^{k-1}x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y)
\end{aligned}$$

Hence

$$\begin{aligned}
A &= (\mathrm{T}_-^{n+1,k+1}\,x *_i ((\mathrm{T}_-^{n+1,k+1}\,y *_i [\mathrm{d}_+^k y]) *_{i+1} [\mathrm{d}_+^{k-1}y])) *_{i+1} \\
&\quad ([\mathrm{d}_+^{k-1}x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) \\
&= (\mathrm{T}_-^{n+1,k+1}\,x *_i \mathrm{T}_-^{n+1,k+1}\,y *_i [\mathrm{d}_+^k y]) *_{i+1} \\
&\quad (\mathrm{S}_-^{i+1}\,\mathrm{d}_+^{k-1}x *_i [\mathrm{d}_+^{k-1}y]) *_{i+1} ([\mathrm{d}_+^{k-1}x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) \\
&= \mathrm{T}_-^{n+1,k+1}\,z *_i [\mathrm{d}_+^k z] *_{i+1} [\mathrm{d}_+^{k-1}z] \\
&= \mathrm{T}_-^{n+1,k-1}\,z
\end{aligned}$$

by (45) and (50). We get the result in case $l = k-1$. Suppose now the relation holds for an index $l$ such that $1 < l \leq k-1$, we must evaluate

$$B = (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{T}_-^{n+1,l-1}\,y) *_{i+1} (\mathrm{T}_-^{n+1,l-1}\,x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y)$$

Repeated applications of the exchange rule give:

$$\begin{aligned}
B &= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i (\mathrm{T}_-^{n+1,l}\,y *_{n+1-l+1} [\mathrm{d}_+^{l-1}y])) *_{i+1} \\
&\quad ((\mathrm{T}_-^{n+1,l}\,x *_{n+1-l+1} [\mathrm{d}_+^{l-1}x]) *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) \\
&= ((\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{T}_-^{n+1,l}\,y) *_{n+1-l+1} (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i [\mathrm{d}_+^{l-1}y])) *_{i+1} \\
&\quad ((\mathrm{T}_-^{n+1,l}\,x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) *_{n+1-l+1} ([\mathrm{d}_+^{l-1}x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_-^{k-1}y)) \\
&= ((\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{T}_-^{n+1,l}\,y) *_{i+1} (\mathrm{T}_-^{n+1,l}\,x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y)) *_{n+1-l+1} \\
&\quad ((\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i [\mathrm{d}_+^{l-1}y]) *_{i+1} ([\mathrm{d}_+^{l-1}x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_-^{k-1}y)) \\
&= \mathrm{T}_-^{n+1,l}\,z *_{n+1-l+1} [\mathrm{d}_+^{l-1}z] \\
&= \mathrm{T}_-^{n+1,l-1}\,z
\end{aligned}$$

which is the result for $l-1$.

By symmetry, the relations on $\mathrm{T}_+^{n+1,l}\,z$ hold as well. ∎

A.9. Properties of the composition. We now check that the composition of cells we just defined satisfies all the axioms of $(n+1)$-categories.

A.9.1. Associativity. Taking first $x$, $y$ and $z$ as in the previous section, and $i \in \{0, \ldots, n\}$:

$$\mathrm{S}_{-}^{i+1}\,\mathrm{d}_{-}^{k-1}x *_i \mathrm{S}_{-}^{i+1}\,\mathrm{d}_{-}^{k-1}y \;=\; \mathrm{S}_{-}^{i+1}\,\mathrm{d}_{-}^{k-1}z \tag{51}$$
$$\mathrm{S}_{+}^{i+1}\,\mathrm{d}_{-}^{k-1}x *_i \mathrm{S}_{+}^{i+1}\,\mathrm{d}_{-}^{k-1}y \;=\; \mathrm{S}_{+}^{i+1}\,\mathrm{d}_{-}^{k-1}z \tag{52}$$

Just notice that for all $j \in \{k+1, \ldots, n+1\}$,

$$[\mathrm{d}_{+}^{j}x] = [\mathrm{d}_{+}^{j}y] = [\mathrm{d}_{+}^{j}z]$$

and are all identities in the expressions of (51). Repeated applications of the exchange rule plus (46) give the result, and likewise for (52). Now the associativity of $*_i$ on $(HX)_{n+1}$ directly follows from (51), (52) and the composition formula (47).

A.9.2. Identities. We now verify that the cells $\mathrm{I}x$ actually behave like identities. Let $i \leq n$ and $k = n+1-k$. Let $x$ be a cell in $(HX)_i$, $y$ a cell in $(HX)_{n+1}$, such that $\mathrm{d}_{-}^{k}y = x$. Thus $\mathrm{d}_{+}^{k}\mathrm{I}^{k}x = x = \mathrm{d}_{-}^{k}y$, and $\mathrm{I}^{k}x$, $y$ are $i$-composable.

Let $z = \mathrm{I}^{k}x *_i y$. We have to show that $z = y$. But both cells clearly agree on their first four components, so that we only need to prove $[z] = [y]$. Now by (47),

$$[z] = (\mathrm{S}_{-}^{i+1}\,\mathrm{d}_{-}^{k-1}\mathrm{I}^{k}x *_i [y]) *_{i+1} ([\mathrm{I}^{k}x] *_i \mathrm{S}_{+}^{i+1}\,\mathrm{d}_{+}^{k-1}y)$$

With $A = \mathrm{S}_{-}^{i+1}\,\mathrm{d}_{-}^{k-1}\mathrm{I}^{k}x$ and $B = [\mathrm{I}^{k}x] *_i \mathrm{S}_{+}^{i+1}\,\mathrm{d}_{+}^{k-1}y$,

$$[z] = (A *_i [y]) *_{i+1} B \tag{53}$$

We first evaluate $A$; formally:

$$\begin{aligned} A &= \mathrm{S}_{-}^{i+1}\,\mathrm{d}_{-}^{k-1}\mathrm{I}^{k}x \\ &= \mathrm{S}_{-}^{i+1}\,\mathrm{I}x \\ &= \mathrm{a}_{+}^{i+1}\,\mathrm{I}x *_0 [\mathrm{d}_{+}^{i+1}\mathrm{I}x] *_1 \ldots *_{i-1} [\mathrm{d}_{+}^{2}\mathrm{I}x] \\ &= \mathrm{a}_{+}^{i}\,x *_0 [\mathrm{d}_{+}^{i}x] *_1 \ldots *_{i-1} [\mathrm{d}_{+}x] \\ &= \mathrm{T}_{-}^{i}\,x \end{aligned}$$

whose value in (53) is the $(n+2)$-identity on $\mathrm{T}_{-}^{i}\,x$. On the other hand

$$\mathrm{d}_{-}^{k+1}[y] = \mathrm{d}_{-}^{k}\mathrm{d}_{-}[y] = \mathrm{d}_{-}^{k}\,\mathrm{T}_{-}^{n+1}\,y = \mathrm{T}_{-}^{n-k+1}\,\mathrm{d}_{-}^{k}y = \mathrm{T}_{-}^{i}\,x$$

Hence $A *_i [y] = [y]$, and $[z]$ becomes $[y] *_{i+1} B$.

Let us evaluate $B$:

$$\begin{aligned} B &= [\mathrm{I}^{k}x] *_i \mathrm{S}_{+}^{i+1}\,\mathrm{d}_{+}^{k-1}y \\ &= [x] *_i [\mathrm{d}_{-}^{2}\,\mathrm{d}_{+}^{k-1}y] *_{i-1} \ldots *_1 [\mathrm{d}_{-}^{i+1}\mathrm{d}_{+}^{k-1}y] *_0 \mathrm{a}_{-}^{i+1}\,\mathrm{d}_{+}^{k-1}y \\ &= [x] *_i [\mathrm{d}_{-}x] *_{i-1} \ldots *_1 [\mathrm{d}_{-}^{i}x] *_0 \mathrm{a}_{-}^{i+1}\,\mathrm{d}_{+}^{k-1}y \end{aligned}$$

which denotes in fact an $(n+2)$-identity in (53). On the other hand,

$$\begin{array}{rcl} \mathrm{d}_+^k[y] & = & \mathrm{d}_+^{k-1}\mathrm{d}_+[y] \\ & = & \mathrm{d}_+^{k-1}\,\mathrm{T}_+^{n+1}\,y \\ & = & \mathrm{T}_+^{i+1}\,\mathrm{d}_+^{k-1}y \\ & = & [\mathrm{d}_-\mathrm{d}_+^{k-1}y] *_i \ldots [\mathrm{d}_-^{i+1}\mathrm{d}_+^{k-1}y] *_0 \mathrm{a}_-^{i+1}\,\mathrm{d}_+^{k-1}y \\ & = & [x] *_i [\mathrm{d}_-x] *_{i-1} \ldots *_1 [\mathrm{d}_-^i x] *_0 \mathrm{a}_-^{i+1}\,\mathrm{d}_+^{k-1}y \end{array}$$

so that $B$ is the $(n+2)$-identity on $\mathrm{d}_+^k[y]$, whence $[y] *_{i+1} B = [y]$, and we get the desired result. Of course the same holds for identities on the right.

We finally show that for $i$-composable $n$-cells $x^n$, $y^n$ in $(HX)_n$,

$$\mathrm{I}(x^n *_i y^n) = \mathrm{I}x^n *_i \mathrm{I}y^n$$

Again the equality is obvious on the first four components. As regards the last component, taking $l = n - i$ and applying (45) gives:

$$\begin{array}{rcl} [\mathrm{I}(x^n *_i y^n)] & = & \mathrm{id}^{n+2}([x^n *_i y^n]) \\ & = & \mathrm{id}^{n+2}((\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{l-1}x^n *_i [y^n]) *_{i+1} ([x^n] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{l-1}y^n)) \\ & = & (\mathrm{id}^{n+2}(\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{l-1}x^n) *_i \mathrm{id}^{n+2}([y^n])) *_{i+1} \\ & & (\mathrm{id}^{n+2}([x^n]) *_i \mathrm{id}^{n+2}(\mathrm{S}_+^{i+1}\,\mathrm{d}_+^{l-1}y^n)) \\ & = & (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{l-1}\mathrm{id}^{n+1}(x^n) *_i [\mathrm{I}y^n]) *_{i+1} \\ & & ([\mathrm{I}x^n] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{l-1}\mathrm{id}^{n+1}(y^n)) \\ & = & [\mathrm{I}x^n *_i \mathrm{I}y^n] \end{array}$$

A.9.3. Exchange. We prove here that $HX$ satisfies the exchange rule.

Let $x$, $y$, $z$, $t$ be cells in $(HX)_{n+1}$, and $0 \leq i < j < n+1$. Define $k = n+1-i$ and $l = n+1-j$. We suppose that

$$\begin{array}{rcl} \mathrm{d}_+^k x & = & \mathrm{d}_+^k z = \mathrm{d}_-^k y = \mathrm{d}_-^k t \\ \mathrm{d}_+^l x & = & \mathrm{d}_-^l z \\ \mathrm{d}_+^l y & = & \mathrm{d}_-^l t \end{array}$$

such that the following composites are well defined in $(HX)_{n+1}$:

$$\begin{array}{rcl} A & = & (x *_i y) *_j (z *_i t) \\ B & = & (x *_j z) *_i (y *_j t) \end{array}$$

and we have to prove that $A = B$. Here again the equality on the first four components is easy: it remains to prove $[A] = [B]$.

First

$$[B] = (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}(x *_j z) *_i [y *_j t]) *_{i+1} ([x *_j z] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}(y *_j t)) \tag{54}$$

Notice that $\mathrm{d}_-^{k-1}(x *_j z) = \mathrm{d}_-^{k-1}x$ and $\mathrm{d}_+^{k-1}(y *_j t) = \mathrm{d}_+^{k-1}t$, and these are identities in (54). By using the exchange rule and (47), $[B]$ rewrites in the form:

$$(T *_{j+1} Y) *_{i+1} (Z *_{j+1} X) \tag{55}$$

where

$$\begin{array}{rcl} X & = & ([x] *_j \mathrm{S}_+^{j+1}\,\mathrm{d}_+^{l-1}z) *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{l-1}t \\ Y & = & \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i ([y] *_j \mathrm{S}_+^{j+1}\,\mathrm{d}_+^{l-1}t) \\ Z & = & (\mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}x *_j [z]) *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t \\ T & = & \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i (\mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}j *_j [t]) \end{array}$$

But $i+1 < j+1$ so that the exchange rule applies and we get

$$[B] = (T *_{i+1} Z) *_{j+1} (Y *_{i+1} X) \tag{56}$$

Let us evaluate $T *_{i+1} Z$: by distributing the identities,

$$T *_{i+1} Z = (U *_j U') *_{i+1} (V *_j V') \tag{57}$$

where

$$\begin{array}{rcl} U & = & \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}y \\ U' & = & \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i [t] \\ V & = & \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t \\ V' & = & [z] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t \end{array}$$

Here the argument splits in two cases:

$\boxed{\text{Case 1}}$. Suppose $i+1 < j$.

By applying exchange to (57):

$$T *_{i+1} Z = (U *_{i+1} V) *_{j+1} (U' *_{i+1} V') \tag{58}$$

First

$$\begin{array}{rcl} U' *_{i+1} V' & = & (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i [t]) *_{i+1} ([z] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t) \\ & = & (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}z *_i [t]) *_{i+1} ([z] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t) \\ & = & [z *_i t] \end{array}$$

because $\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x = \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}z$ On the other hand

$$\begin{aligned} U *_{i+1} V &= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}y) *_{i+1} (\mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t) \\ &= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}y) *_{i+1} (\mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y) \\ &= \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}(x *_i y) \end{aligned}$$

because $\mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t = \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}y$, and the last step is a particular case of Lemma A.8.

Now

$$T *_{i+1} Z = \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}(x *_i y) *_j [z *_i t]$$

and of course the same arguments show that

$$Y *_{i+1} X = [x *_i y] *_j \mathrm{S}_+^{j+1}\,\mathrm{d}_+^{l-1}(z *_i t)$$

such that

$$[B] = [(x *_i y) *_j (z *_i t)] = [A]$$

$\boxed{\text{Case 2}}$. Suppose $i + 1 = j$.

Here

$$\begin{aligned} T *_{i+1} Z &= T *_j Z \\ &= (U *_j U') *_j (V *_j V') \\ &= U *_j (U' *_j V) *_j V' \end{aligned}$$

and we rewrite $U' *_j V$ as $\tilde{V} *_j \tilde{U'}$ where

$$\begin{aligned} \tilde{U'} &= \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}z *_i [t] \\ \tilde{V} &= \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i \mathrm{T}_-^{j}\,\mathrm{d}_-^{l}t \end{aligned}$$

In fact

$$\begin{aligned} V &= \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}x *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t \\ &= \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i [\mathrm{d}_+^{l+1}x] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t \\ &= \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i \mathrm{T}_+^{i+1}\,\mathrm{d}_+^{k-1}t \end{aligned}$$

but $\mathrm{d}_-^{l}\,\mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x = \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x$ and $\mathrm{d}_+^{l}[t] = \mathrm{T}_+^{i+1}\,\mathrm{d}_+^{k-1}t$ such that exchange may be applied to $U' *_j V$, giving

$$\begin{aligned} U' *_j V &= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i [t]) *_j (\mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i \mathrm{T}_+^{i+1}\,\mathrm{d}_+^{k-1}t) \\ &= \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i [t] \\ &= (\mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i \mathrm{T}_-^{j}\,\mathrm{d}_-^{l}t) *_j (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}z *_i [t]) \end{aligned}$$

The last equality comes from

$$\mathrm{d}_-^{l+1}[t] = \mathrm{T}_-^{j}\,\mathrm{d}_-^{l}t$$

and

$$\mathrm{d}_+^{l+1}\,\mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x = \mathrm{S}_-^{i+1}\,\mathrm{d}_+^{l}x = \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}z$$

Now

$$\begin{aligned} \tilde{U'} *_j V' &= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}z *_i [t]) *_j ([z] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t) \\ &= (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}z *_i [t]) *_{i+1} ([z] *_i \mathrm{S}_+^{i+1}\,\mathrm{d}_+^{k-1}t) \end{aligned}$$

This proves

$$\tilde{U'} *_j V' = [z *_i t] \tag{59}$$

Finally

$$U *_j \tilde{V} = (\mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x *_i \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}y) *_j (\mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i \mathrm{T}_-^{j}\,\mathrm{d}_-^{l}t) \tag{60}$$

But (37) gives

$$\begin{aligned} \mathrm{d}_-^{l+1}\,\mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x &= \mathrm{d}_-\,\mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x \\ &= \mathrm{T}_-^{j,2}\,\mathrm{d}_-^{k-1}x \\ &= \mathrm{S}_-^{i+1}\,\mathrm{d}_-^{k-1}x \end{aligned}$$

and also

$$\begin{aligned} \mathrm{d}_+^{l+1}\,\mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}y &= \mathrm{d}_+\,\mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}y \\ &= \mathrm{d}_+\,\mathrm{T}_-^{j+1,2}\,\mathrm{d}_-^{l-1}y \\ &= \mathrm{T}_-^{j}\,\mathrm{d}_+^{l}y \\ &= \mathrm{T}_-^{j}\,\mathrm{d}_-^{l}t \end{aligned}$$

Thus exchange may be applied to (60), so that

$$\begin{aligned} U *_j \tilde{V} &= \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}y \\ &= \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}x *_i \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}y *_i [\mathrm{d}_+^{l+1}y] \\ &= \mathrm{T}_-^{j+1,3}\,\mathrm{d}_-^{l-1}(x *_i y) *_i [\mathrm{d}_+^{l+1}y] \end{aligned}$$

because of (50) and finally

$$U *_j \tilde{V} = \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}(x *_i y) \tag{61}$$

by using

$$\mathrm{d}_+^{l+1}y = \mathrm{d}_+^{l+1}(x *_i y)$$

Now (59) and (61) show that

$$\begin{aligned} T *_{i+1} Z &= U *_j (U' *_j V) *_j V' \\ &= U *_j (\tilde{V} *_j \tilde{U'}) *_j V' \\ &= (U *_j \tilde{V}) *_j (\tilde{U'} *_j V') \\ &= \mathrm{S}_-^{j+1}\,\mathrm{d}_-^{l-1}(x *_i y) *_{i+1} [z *_i t] \end{aligned}$$

By symmetry

$$Y *_{i+1} X = [x *_i y] *_j \mathrm{S}_+^{j+1} \mathrm{d}_+^{l-1}(z *_i t)$$

so that in this case again

$$[B] = [(x *_i y) *_j (z *_i t)] = [A]$$

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